B.A. WESTWOOD

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Preface

Even after over fifty years, Einstein's theory of Relativity still has the reputation of being difficult to understand. If this is true it is not because of mathematical difficulties – this book rarely uses anything more complicated than a square root sign – but because Relativity involves us in thinking very hard about the making and interpretation of measurements in physics. This makes it all the more useful for students of physics to study Relativity, even though, unless they work in a few special fields, they are unlikely ever to observe relativistic effects in practice.

This book is based on lectures given to first-year undergraduates who are not physics specialists. It is intended as an introduction to the subject both for sixth forms in schools and for university students. Exercises have been included in the text to illuminate important points of the discussion; I hope these and the exercises at the end of each chapter will help readers to understand the formulae of relativistic physics by using them. Two books may be recommended for further reading – they both discuss the topics covered here at greater length than was possible in a book of this size. They are:

J. H. Smith, *Introduction to Special Relativity* (Benjamin)
French, *Special Relativity* (Nelson)

B. A. WESTWOOD
Churchill College, Cambridge

Contents

Introduction

Einstein produced his theory of Relativity between 1905 and 1915. Difficulties had arisen in combining the old and well-established results of Newtonian mechanics with the newer predictions of electromagnetism, and Einstein was the first to see that the solution lay in modifying fundamental concepts about space and time.

From considering how the same events appear to two observers in relative motion (hence the name) the theory moves on to such surprising results as the relation between mass and energy ($E = mc^2$, the most famous equation in physics), and the curvature of space-time. If the relative motion is just a constant velocity, the theory, then called Special Relativity, is very much simpler. We shall deal mostly with this special case, but will discuss the General Theory, which includes relative accelerations, in the last chapter.

When Einstein first produced his theory, its predictions were far removed from normal experience and difficult to test experimentally. Today, however, many people, including nuclear power engineers and elementary particle physicists, deal with relativistic effects every day as part of their jobs. This book often uses space pilots as examples of observers in hypothetical experiments. Real astronauts have yet to travel faster than one-thousandth of the speed of light, so that the effects of relativity are not obvious to them, but it cannot be long before the Twin Paradox of Chapter 3 is a commonplace result of interstellar travel.

The Michelson Morley experiment

1.1 Electromagnetic waves and the ether

In 1865, Maxwell predicted the existence of electromagnetic waves. Starting from the fundamental equations of electricity and magnetism, he was able to show that an oscillating charge should send out waves. These waves consist of oscillating electric and magnetic fields, which are usually at right angles to each other and to the direction of propagation. Their existence was confirmed by Hertz in 1888. Visible light is only a small part of the electromagnetic spectrum – those waves with wavelengths between 4 and 7×10^{-7}m – but we shall use the term 'light' for any kind of electromagnetic radiation, e.g. radio waves or X-rays. The velocity of light in a vacuum, *c*, will turn out to be a very important quantity; its numerical value is very nearly 3×10^8m/s.

At the time when electromagnetic waves were discovered, all known types of wave (e.g. sound waves, water waves) travelled in some kind of material medium (air, water). It was therefore natural to assume that electromagnetic waves also had their medium, provisionally named the ether, and attempts were made to detect it. But the ether proved very elusive, although it must be present everywhere where light can be transmitted, even in a vacuum or in interstellar space. All attempts to measure its density or elastic modulus, or indeed any other property, failed to obtain positive results.

1.2 The velocity of the earth through the ether

Whatever the properties of the ether, it should be possible to measure the velocity between it and the earth. Consider the experiment outlined in Fig. 1, in which light is reflected from

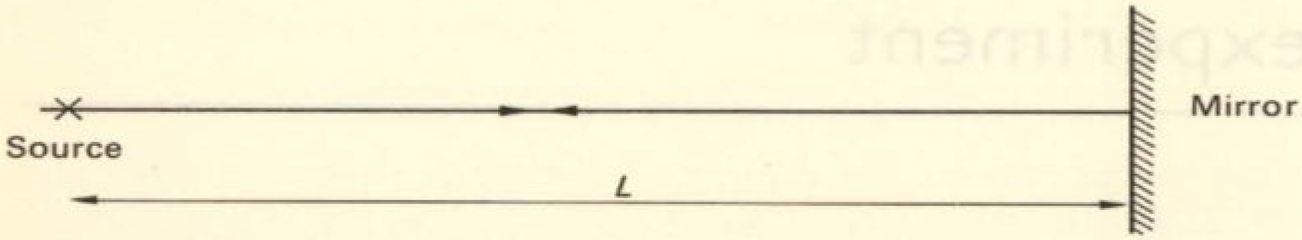


Fig. 1

a distant mirror and the time taken for the round trip is measured. (A rotating mirror might be used, as in Fizeau's experiment, or more modern equipment involving Kerr cells, electric shutters, and electronic gadgetry.) Since the ether is the medium in which light waves propagate, the velocity of light with respect to the ether should always be c, and if the earth is at rest in the ether, the measured round trip time will be $2L/c$. But if the earth (and the apparatus fixed to it) is moving through the ether with a component of velocity v in the direction source-mirror, the outward velocity of the light with respect to the earth will be $c - v$ and the inward velocity $c + v$, giving a total time of

$$\frac{L}{c-v} + \frac{L}{c+v} = \frac{2Lc}{c^2 - v^2}.$$

This difference in times can be used to calculate v, or rather, since we can never know the exact time when the earth is at rest in the ether, to detect changes in v. If the ether is fixed with respect to the sun (which is as good a guess as any other), the component of the earth's velocity through the ether which lies in the source-mirror direction must change during the year by quantities of the order of magnitude of the earth's orbital velocity round the sun, which is 30 km/s or $10^{-4}c$. The round trip

times for occasions when $v = 10^{-4}c$ and 0 respectively will differ by

$$\Delta t = \frac{2L}{c}\left\{\frac{1}{\left(1 - \frac{v^2}{c^2}\right)} - 1\right\}$$

which may be approximated as

$$\frac{2L}{c} \cdot \frac{v^2}{c^2} \text{ if } v \ll c.$$

The effect is very small. The longest path L ever used in this type of experiment was the 38 km from Mt. Wilson to Mt. San Antonio, California, in Michelson's 1924 measurement of the velocity of light. Putting this in the formula gives $\Delta t = 2.5 \times 10^{-12}$ s, which is too small to measure directly even today. In the 1880s, the available light paths were shorter and the clocks less accurate, so it appeared that the velocity of the earth with respect to the ether could be measured in principle but not in practice.

1.3 The Michelson Morley experiment

Fortunately more indirect methods were also capable of detecting the earth's motion through the ether. Suppose that one beam of light is split into two which are sent along different paths and then recombined. Interference fringes will be formed, and any difference in the times taken to traverse the two paths will show up as a shift in the position of the dark and bright fringes. A time difference Δt is equivalent to a path difference of $c\Delta t$, and will thus shift the pattern by $c\Delta t/\lambda \times$ (width of one fringe), where λ is the wavelength of the light. With practice one can easily detect a shift of 0·02 fringes, which for $\lambda = 6 \times 10^{-7}$ m corresponds to a time difference of 4×10^{-17} s! With such sensitivity experiments to measure v are difficult but no longer impossible.

In 1881 Michelson set up the experiment shown diagrammatically in Fig. 2. Light from the sources is split into two beams by the half-silvered mirror A. After reflection from mirrors

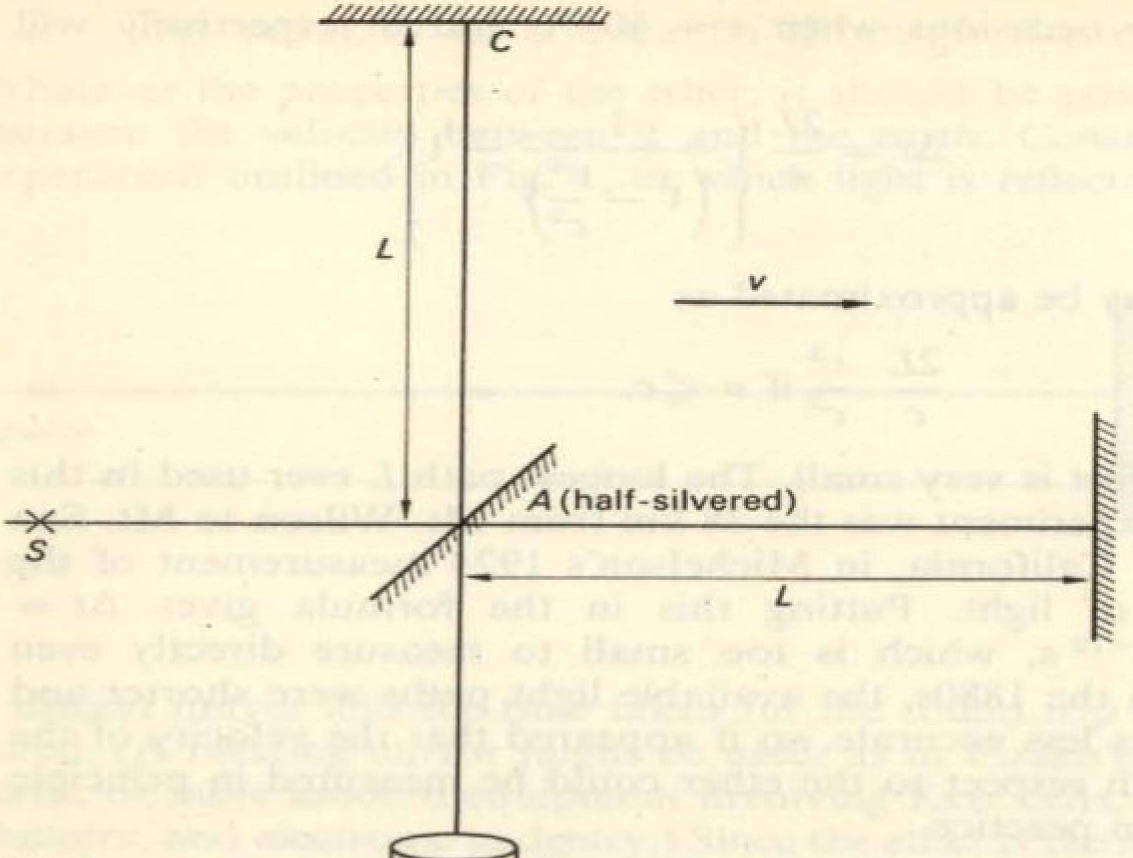


Fig. 2

B and C, the beams are recombined and produce fringes which are viewed through the eyepiece D. (B and C are not quite perpendicular, producing parallel fringes, as with a wedge.) The arms AB and AC are of equal length L.

Suppose the earth is moving through the ether with velocity v in the direction AB. The round trip time for light following path ABA is

$$\frac{2Lc}{c^2 - v^2},$$

as calculated in the previous section. The light travelling from A to C must have a component of velocity v along AB, relative to the ether, or it will fail to strike the mirror at C. (The effect is the same as swimming across a river with a strong current: it is essential to aim at a point considerably upstream of where you wish to land.) Since the total velocity of the light relative to the ether is c, subtracting this component leaves a velocity of

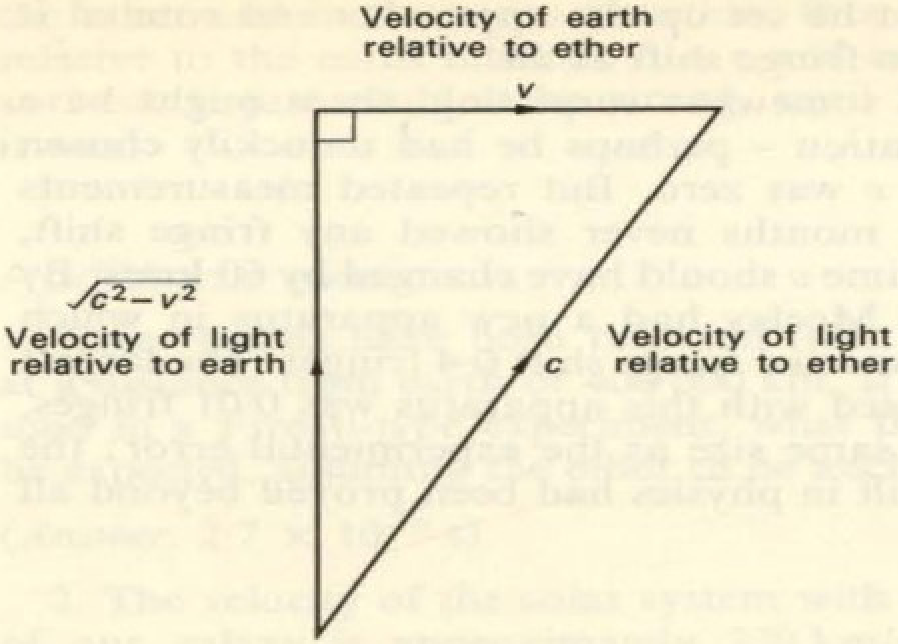


Fig. 3

$\sqrt{(c^2 - v^2)}$ along AC (see Fig. 3). The same is true for the return journey, so the total time for path ACA is

$$\frac{2L}{\sqrt{c^2 - v^2}}.$$

The time difference between the two beams is

$$\Delta t = \frac{2L}{c}\left\{\frac{1}{\left(1 - \frac{v^2}{c^2}\right)} - \frac{1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}\right\} \approx \frac{L}{c} \cdot \frac{v^2}{c^2}$$

since $v \ll c$, corresponding to a fringe shift

$$\Delta f = \frac{c\Delta t}{\lambda} = \frac{L}{\lambda} \cdot \frac{v^2}{c^2}.$$

The easiest way of measuring the fringe shift is to rotate the whole apparatus through 90°. Since this interchanges arms AB and AC and reverses the sign of the fringe shift, an overall shift of $2\Delta f$ should be observed. Michelson's first apparatus had arms 1·2 m long; with $\lambda = 6 \times 10^{-7}$ m and $v = 10^{-4}c$, he expected a shift of 0·04 fringes, or twice the least shift he could

detect. However when he set up the apparatus and rotated it carefully, there was no fringe shift at all!

Although this was somewhat surprising, there might have been a simple explanation – perhaps he had unluckily chosen a time of year when v was zero. But repeated measurements over a period of six months never showed any fringe shift, although during that time v should have changed by 60 km/s. By 1887, Michelson and Morley had a new apparatus in which L was 11 m and the expected fringe shift 0·4 fringes. The largest fringe shift ever detected with this apparatus was 0·01 fringes, which was about the same size as the experimental error: the most famous null result in physics had been proved beyond all reasonable doubt.

1.4 Explanations of the null result

Many attempts were made to explain the non-existence of the fringe shift. One was to suppose that the ether near the earth is dragged along with it: If this were true any experiment carried out wholly on the earth would always show it to be apparently at rest in the ether. But astronomical evidence was available to prove that ether drag did not exist.

Another explanation, proposed by Fitzgerald and shown by Lorentz to follow from the electromagnetic theory of matter, was for all lengths parallel to the direction of the earth's motion through the ether to be contracted by a factor $\sqrt{(1 - v^2/c^2)}$. This would explain the Michelson-Morley result for apparatus with arms of equal length, but would predict a fringe shift on rotating an apparatus with unequal arms. (**Exercise.** Prove this using the equations of the previous section.) Kennedy and Thorndike (1932) used an unequal arms apparatus but failed to detect any shift.

Einstein's solution was simple: the velocity of light with respect to the earth is always c, whether the earth moves through the ether or not. This obviously explains the Michelson-Morley result as the round trip time is always $2L/c$ for any path, but makes nonsense of the addition of velocities. If the velocity of light is c and the velocity of the earth is v, both relative to

the ether and in the same direction, then the velocity of light relative to the earth must be $c - v$ and cannot be c. This was so basic it just couldn't be wrong, until Einstein proved that it was!

Additional exercises

1. Laser beams have been reflected from the moon's surface, at a distance from earth of 400 000 km. If this light path were used in a Fizeau-type experiment, what time difference would be expected, assuming the ether to be anchored to the sun?

(*Answer*. $2{\cdot}7 \times 10^{-8}$ s)

2. The velocity of the solar system with respect to the centre of our galaxy is approximately 220 km/s. If the ether were anchored to the centre of the galaxy, what fringe shift would Michelson have obtained with his original apparatus, as described in the text?

(*Answer*. 2·15 fringes)

3. Suppose the velocity v in Fig. 2 is at an angle θ to the arm AB. Show that the time difference varies with the angle θ approximately as $\Delta t = \frac{v^2}{c^2} \cdot \frac{L}{c} \cdot \cos 2\theta$ and hence that if $\theta = 45°$ there will be no shift of fringes when the apparatus is rotated through 90°.

2

Einstein's postulates

To understand Einstein's statement of the principles of Relativity we must first discuss some of the concepts involved, in the context of classical physics.

2.1 Coordinate systems

A coordinate system provides a way of identifying a particular point in space. This is usually done by taking three mutually perpendicular lines as axes and specifying coordinates x, y, z as the perpendicular distances from the point to the corresponding axis. (An alternative name for 'coordinate system' is 'frame of reference'.) The point where the axes meet is the origin O of the coordinate system and the coordinates of a point are usually written (x, y, z). The coordinate system as a whole is denoted by S.

Relativity is concerned with how the same events appear when viewed in different coordinate systems. Let us compare events as seen in S and in a second coordinate system S', with origin at O' and coordinates x', y' and z'. In general the behaviour of the same object may appear to be quite different in S and in S'. For instance, when we do experiments in the laboratory we usually assume that our workbench is fixed and take some point on it as the origin of our measurements. But in a coordinate system fixed to the sun, the bench (and indeed the whole laboratory) is not stationary but rotates round the origin once a year, not to mention the complicated motion due to the earth's turning on its axis once every 24 hours.

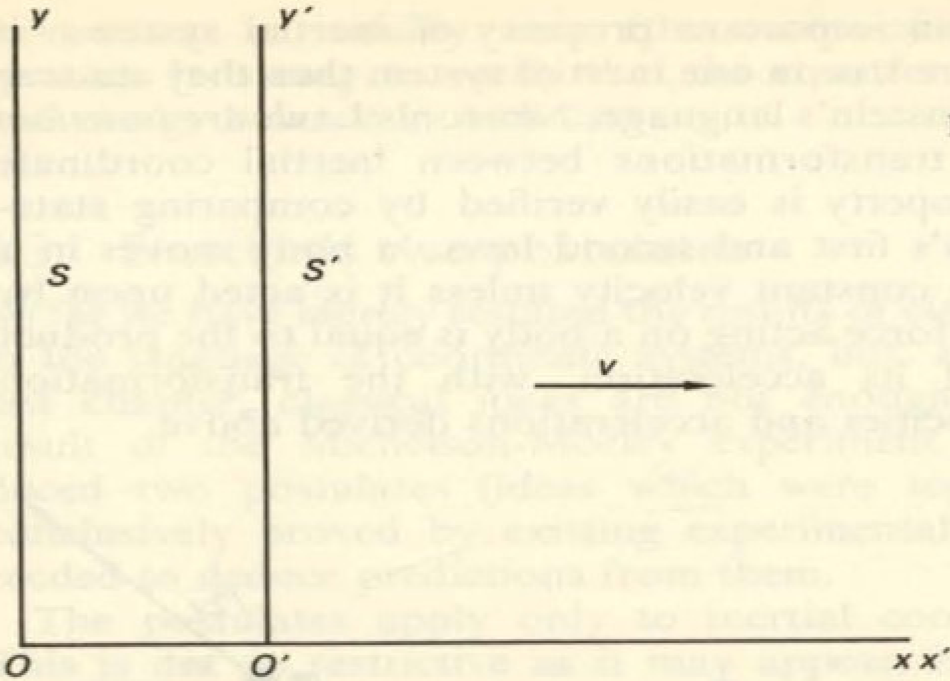


Fig. 4

2.2 Inertial systems

Every thing is much simpler if S and S' have constant relative velocity. Coordinate systems related in this way are called *inertial*, and have special properties. Let us assume that O and O' coincide at time $t = 0$ and that O' then moves along Ox with velocity v, as shown in Fig. 4. Then the two sets of coordinates (x, y, z) and (x', y', z') representing the same point in s and in S' are related by equations

$$x' = x - vt, \; y' = y, \; z' = z.$$

These are usually called the *Gallilean transformation*, after the great precursor of Newton.

Using the Gallilean transformation, we see that the velocity of a body in S' is simply the velocity in S with a velocity v parallel to Ox subtracted

$$\frac{dx'}{dt} = \frac{dx}{dt} - v, \quad \frac{dy'}{dt} = \frac{dy}{dt}, \quad \frac{dz'}{dt} = \frac{dz}{dt}$$

and that the accelerations of the same body in S and in S' are equal

$$\frac{d^2x'}{dt^2} = \frac{d^2x}{dt^2}, \quad \frac{d^2y'}{dt^2} = \frac{d^2y}{dt^2}, \quad \frac{d^2z'}{dt^2} = \frac{d^2z}{dt^2}.$$

This leads to an important property of inertial systems: if Newton's Laws are true in one inertial system then they are true in all, or to use Einstein's language, Newton's Laws are *invariant* under Gallilean transformations between inertial coordinate systems. This property is easily verified by comparing statements of Newton's first and second laws, 'a body moves in a straight line with constant velocity unless it is acted upon by a force', and 'the force acting on a body is equal to the product of its mass and its acceleration' with the transformation properties of velocities and accelerations derived above.

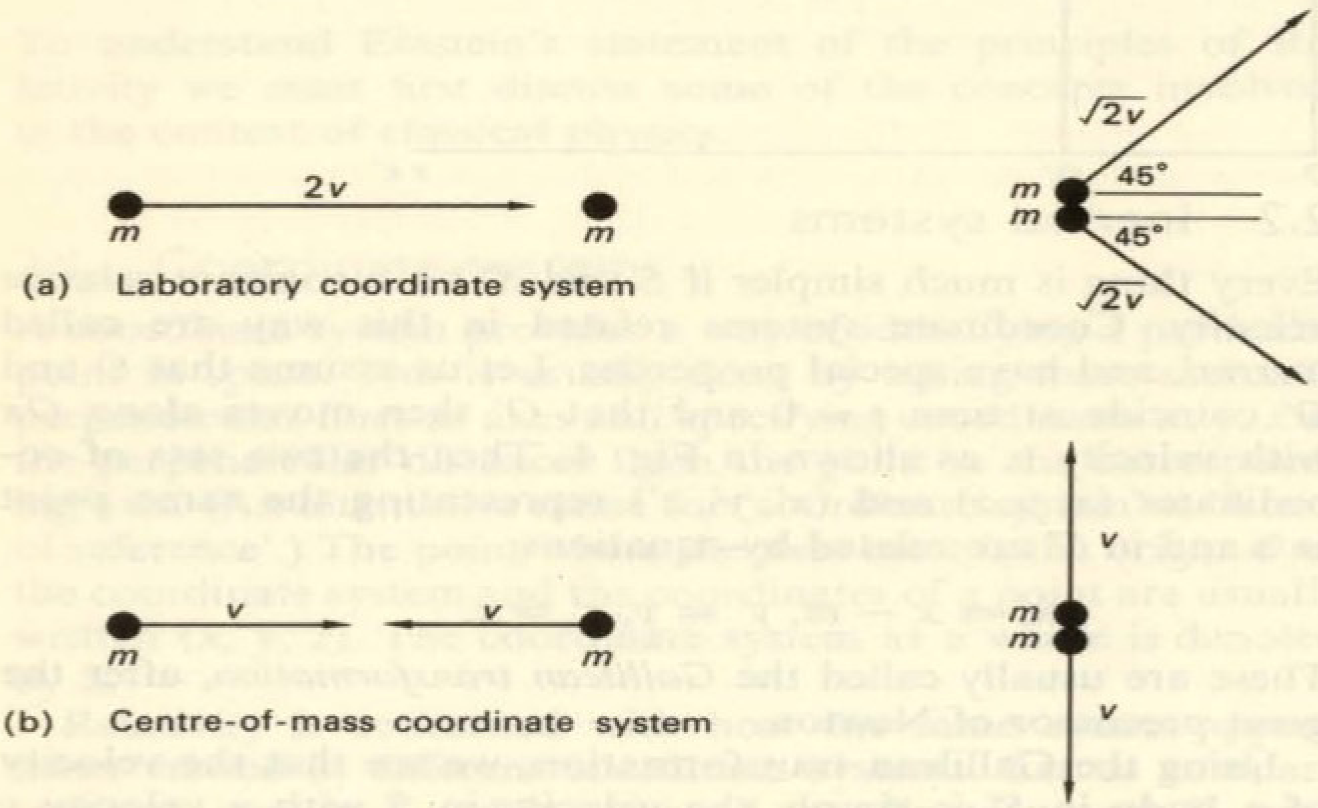


Fig. 5

Alternatively we can consider a simple example: Fig. 5(a) shows the results of applying the conservation laws of energy and momentum, in the laboratory system, to a glancing collision between two billiard balls, while Fig. 5(*b*), in which $-v$ along Ox has been added to all velocities, shows how the collision would appear in a coordinate system in which the total

momentum is initially zero (the centre-of-mass coordinate system). It is easy to verify that the equations of momentum and energy balance in each case.

2.3 Einstein's two postulates

So far we have merely restated the results of classical mechanics in the language of coordinate systems, but, as we saw in the last chapter, classical ideas are not enough to explain the result of the Michelson-Morley experiment. Einstein introduced two postulates (ideas which were suggested but not conclusively proved by existing experimental data) and proceeded to deduce predictions from them.

The postulates apply only to inertial coordinate systems. This is not so restrictive as it may appear, however, because coordinate systems which are not strictly inertial can often be treated as if they were for particular purposes. For example, suppose we have a train running alongside a level embankment and wish to consider how events on the train appear from outside. Because of the rotation of the earth, coordinate systems fixed to train and embankment are not strictly inertial, but the effects involved are so nearly equal that the error introduced by treating both systems as inertial is quite negligible.

In this way we shall find that the Special Theory of Relativity has very wide applications. Some exceptions will be discussed briefly in the chapter on General Relativity.

2.4 The Relativity postulate

Motion at constant velocity in a train or aeroplane does not produce any immediately noticeable effects, apart from the changing view through the windows; walking down the aisle or drinking coffee in a jet plane travelling at several hundred metres per second feels very much like doing the same thing at rest on the ground, unless the plane hits an air pocket and the motion becomes accelerated. Similarly, it is the jolting over the points which spills soup in a railway dining car, not steady motion along the rails.

This kind of 'experiment', together with more quantitative investigations such as the billiard ball collision of Fig. 5, led to Einstein's first postulate. It is usually stated: *It is impossible to perform an experiment to detect absolute motion of an inertial coordinate system.* This means that while relative motion of coordinate systems can be detected (e.g. by looking out of the windows), it is impossible to pick out one special coordinate system as being absolutely 'at rest'.

The postulate can be rephrased: *the laws of physics are the same in all inertial coordinate systems.* This must be so, or absolute motion might be detected after all; for example, by the laws taking a particularly simple form in the system which was absolutely at rest.

2.5 The postulate of the constant velocity of light

Einstein's second postulate states: *the velocity of light is the same in all inertial coordinate systems.* Measurements of the velocity of light should therefore give results inedpendent of the velocity of the source. Terrestrial experiments provide no significant test, but there is some astronomical evidence concerning the time variation of the intensity of light from double stars. These consist of two stars rotating with very high velocities about their common centre of gravity. If the velocity of light relative to an observer did depend on the velocity of the source (as is the case with all other kinds of wave), the intensity would vary in a very different way from that which is actually observed.

This postulate seems simple, but has far-reaching consequences. It immediately explains the result of the Michelson-Morley experiment because, if the velocity of the light is always the same, the round trip time for both arms must always be equal, whatever the velocity of the earth through the ether. In fact this postulate implies that light cannot behave like other kinds of wave, and the idea of a medium for it is therefore meaningless. So this is the last we shall hear of the ether concept.

Additional exercises

1. Suppose the billiard balls in Fig. 5 are replaced by balls of putty which stick together on impact. Show that the kinetic energy lost on impact is the same in laboratory and centre-of-mass coordinate systems.

2. Consider a general elastic collision between two particles and show directly that the classical conservation laws of momentum and kinetic energy for this system are invariant under Gallilean transformations.

3

The Lorentz transformation

We shall now begin to deduce some of the results of applying Einstein's two postulates. Although these results are often at variance with classical physics and everyday experience, we shall see that even the most surprising can be verified in suitable experiments.

3.1 Time dilation

A simple, hypothetical experiment will show us that the postulates lead to unexpected conclusions about time measurements. Imagine that the two coordinate systems S and S' are set up as in Fig. 4. When O and O' coincide, a light pulse is sent off from O, reflected from a mirror M at a perpendicular distance d, and picked up by a photocell P, as shown in Fig. 6. The position of P is arranged so that O' and the light pulse arrive there simultaneously.

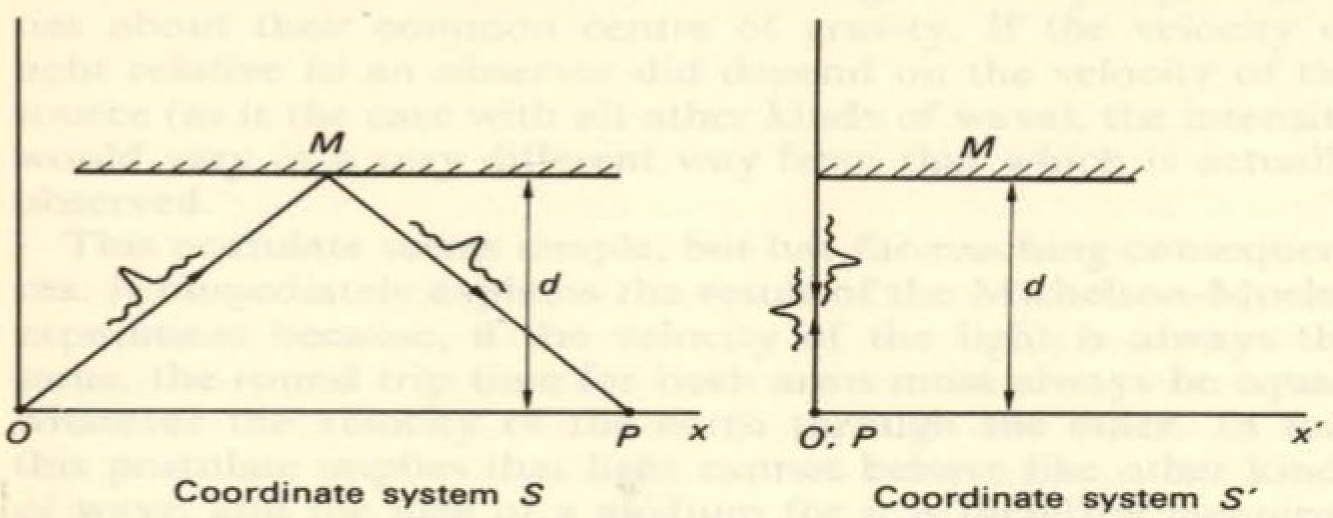


Fig. 6

What time T does an observer at O record for the arrival of the pulse at P? Since OP must be equal to vT for O' to arrive at the same time as the pulse, the total distance travelled by the pulse is

$$2\sqrt{(\tfrac{1}{4}v^2T^2 + d^2)}.$$

But since light always travels at velocity c, this must also be equal to cT. Hence

$$T = \frac{2d/c}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}.$$

As seen in S', the pulse always travels perpendicularly to the mirror, and covers a total distance $2d$. By the second postulate, the velocity of light in S' is also c, making the time of arrival of the pulse

$$T' = \frac{2d}{c}.$$

So observers at O and O' record different values for the time interval between the same two events (departure and arrival of the light pulse)! This effect is called time dilation and is a general result. Time intervals, such as T,' measured on a clock which is present at both events, *proper times*, are always shorter by a factor $\sqrt{(1 - v^2/c^2)}$ than the corresponding times, such as T, in other inertial coordinate systems, *improper times*. This is often expressed more briefly by the phrase, 'moving clocks (which register proper times) run slow'.

3.2 An experimental test of time dilation

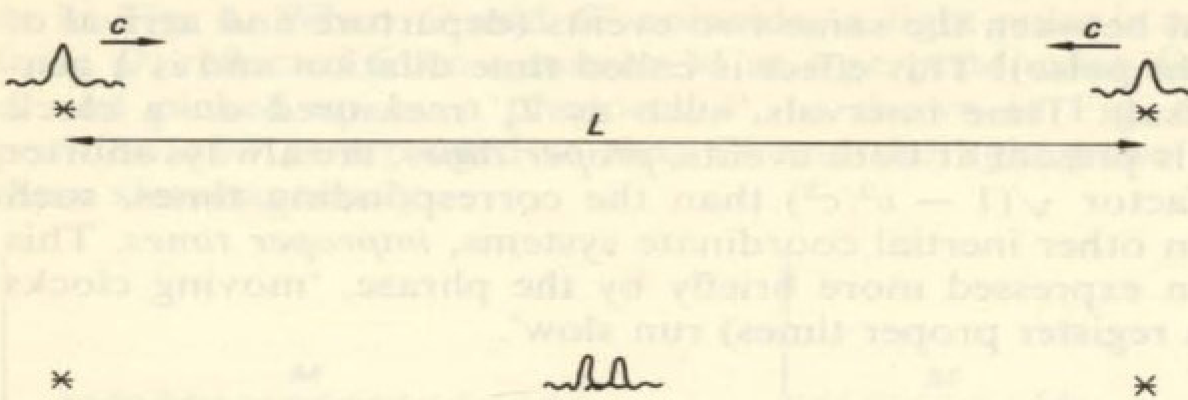
The existence of time dilation can be confirmed in experiments with elementary particles called μ-mesons (or muons for short) which decay spontaneously into an electron, a neutrino and an anti-neutrino. For muons at rest the time taken for half the particles to decay, the half-life, which is a proper time measurement, is $1{\cdot}53 \times 10^{-6}$ s. We can measure the half-life of muons in cosmic rays (which travel at speeds in excess of $0{\cdot}99c$) by

comparing the numbers which arrive each second at the top of a mountain and at sea-level. This improper time measurement gives a value for the half-life of $13{\cdot}5 \times 10^{-6}$ s. We are observing time dilation by a factor of 9. The agreement with the expected factor of $1/\sqrt{(1 - 0{\cdot}99^2)} \approx 7{\cdot}1$ is only approximate, but the experiment is difficult to perform. Much more exact tests of Special Relativity will be discussed later.

3.3 Relativity and simultaneity

In classical physics, two events which occur simultaneously in one coordinate system appear to be simultaneous in all other coordinate systems. Einstein's second postulate upsets this, as we can see from a simple hypothetical example.

Suppose two light sources, one at $x = 0$ and one at $x = L$, each send a light pulse in the direction of the other source. If the two pulses are emitted simultaneously, since each has velocity c, they will meet at $x = \frac{1}{2}L$, halfway between the sources (Fig. 7). We can use this observation, light pulses meeting

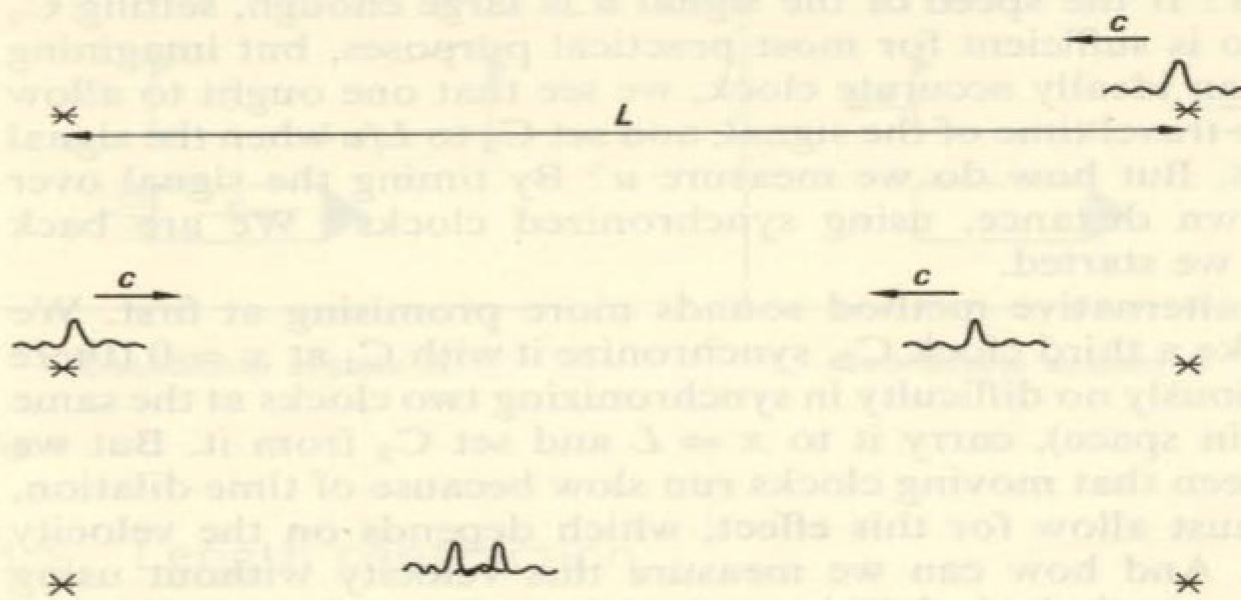


In S : simultaneous emission, pulses meet halfway

Fig. 7

halfway between their sources, as a definition of the simultaneity of the two events, i.e. the emission of the light pulses from their respective sources. It will apply in any inertial coordinate system because, by the second postulate, the velocity of light is the same in all of them.

Let us apply this definition to the emission of the light pulses, as seen in coordinate system S'. We will assume for convenience that O and O' coincide at the time the pulse is emitted from that point. The points $x = L$ and $x' = L$ also coincide at that instant, so if the pulses are also emitted simultaneously in S', they will meet halfway between their points of emission at $x' = \frac{1}{2}L$. This point was opposite $x = \frac{1}{2}L$ when O and O' coincided, but it has moved forward (along with the whole of coordinate system S') by the time the light pulses meet. An observer in S' sees that the pulses meet at a point which is nearer to $x' = 0$ than to $x' = L$ (Fig. 8) and concludes that the



In S': emission not simultaneous; pulses meet nearer to the origin

Fig. 8

pulse from $x' = 0$ was sent some time after the one from $x' = L$. So events which are simultaneous in S are not simultaneous in S'.

3.4 The problem of synchronizing clocks

In the last section we detected the simultaneity of two spatially separated events using light pulses only. A more natural way of detecting simultaneity is to make use of two clocks, one at

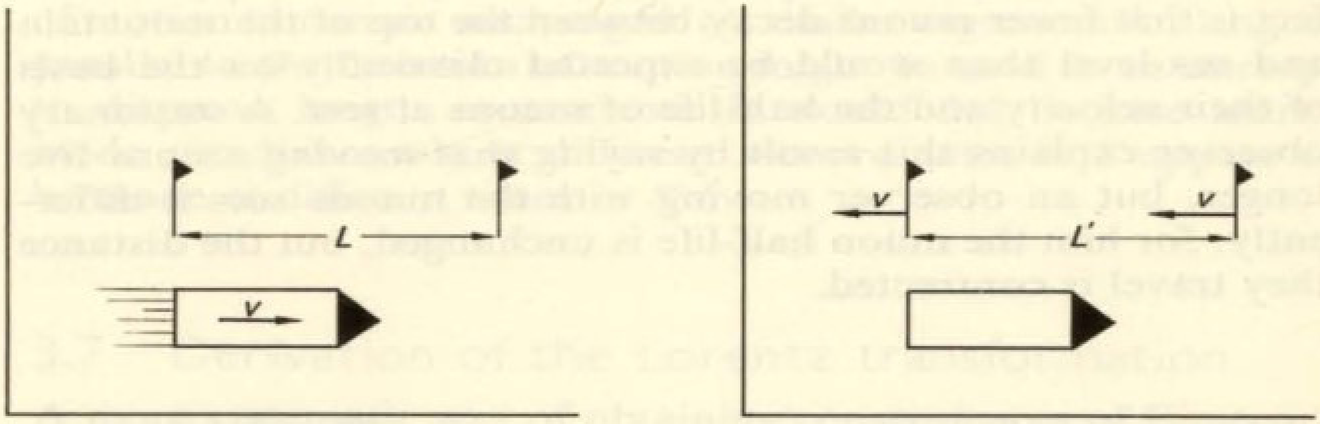
the location of each of the events: if these clocks have been synchronized at some time in the past and now record similar times when two events take place, then the events are simultaneous. It all sounds very simple and obvious, until like Einstein we probe a little more deeply. He showed that the procedure described above is essentially ambiguous, in a way which can only be removed by using the second postulate.

The difficulty is in synchronizing two clocks at a distance, C_1 at $x = 0$ and C_2 at $x = L$ say. Suppose we send some form of signal (other than a light pulse) from C_1 to C_2 at the instant when C_1 is set to zero. How should C_2 be set the when the signal arrives? If the speed of the signal u is large enough, setting C_2 to zero is sufficient for most practical purposes, but imagining C_2 as an ideally accurate clock, we see that one ought to allow for the travel time of the signal, and set C_2 to L/u when the signal arrives. But how do we measure u? By timing the signal over a known distance, using synchronized clocks? We are back where we started.

An alternative method sounds more promising at first. We can take a third clock C_3, synchronize it with C_1 at $x = 0$ (there is obviously no difficulty in synchronizing two clocks at the same point in space), carry it to $x = L$ and set C_2 from it. But we have seen that moving clocks run slow because of time dilation, and must allow for this effect, which depends on the velocity of C_3. And how can we measure this velocity without using synchronized clocks? This method is really no better than the first.

Einstein saw the way out of this difficulty with the use of light signals. The second postulate ensures that these always travel at velocity c in all coordinate systems. Our measurement problems disappear and we can synchronize the clocks simply by setting C_2 to L/c on receipt of a light signal sent out when C_1 was set to zero. This procedure can be repeated for clocks at any point in space with consistent results. (**Exercise.** C_4 at $x = 2L$ can be set using signals from either C_1 or C_2. Show that the two results agree.) Once the clocks have been synchronized in this way our definition of simultaneity is unambiguous, for a single coordinate system. (The events are still not simultaneous in all coordinate systems of course.)

This discussion may have seemed rather dull and abstract: it is however typical of the way in which Einstein clarified our ideas of space and time and emphasises that our deductions must follow directly from his postulates, even if they seem to run counter to our everyday experience.



Coordinate system S Coordinate system S'

Fig. 9

3.5 Length contraction

Time dilation affects our measurements of length. Fig. 9 shows a hypothetical space-ship travelling at velocity v between two markers separated by a distance L in coordinate system S. The rocket takes time

$$\Delta t = \frac{L}{v}$$

to pass between the markers, but this is an improper time measurement. Applying the usual factor, the corresponding time in the coordinate system in which the space-ship is at rest (S') is

$$\Delta t' = \frac{L}{v}\sqrt{\left(1 - \frac{v^2}{c^2}\right)}.$$

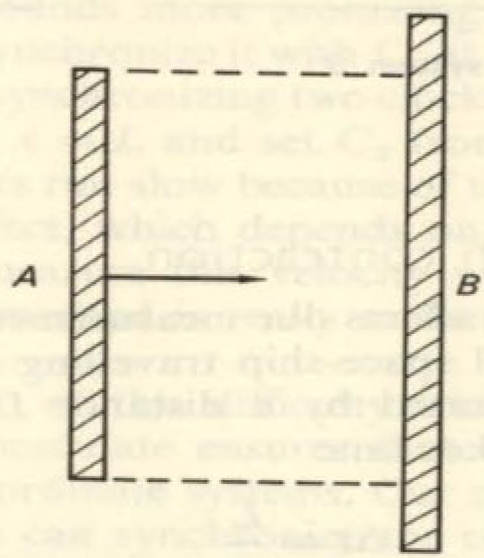
But in this system the markers approach the stationary space-ship with velocity v, and so their distance apart must be

$$L' = v\Delta t' = L\sqrt{\left(1 - \frac{v^2}{c^2}\right)}.$$

Evidently lengths parallel to the direction of motion appear to be shorter in moving coordinate systems.

The muon experiment illustrates the interdependence of time dilation and length contraction very nicely. The experimental fact is that fewer muons decay between the top of the mountain and sea-level than would be expected classically on the basis of their velocity and the half-life of muons at rest. A stationary observer explains this result by saying that moving muons live longer, but an observer moving with the muons sees it differently: for him the muon half-life is unchanged, but the distance they travel is contracted.

Fig. 10



3.6 Perpendicular lengths

Lengths perpendicular to the direction of motion do not change. We can show this by demonstrating that any other result leads to a contradiction of the first postulate. Suppose that in Fig. 10 rod A, moving towards a stationary rod B, does appear to contract, and that we arrange for each rod to make a mark on

the other as they pass, leaving a permanent record of this situation. An observer moving with rod A must interpret these marks as meaning that a moving rod (rod B in this case) appears to expand.

These two contrary views can only be reconciled if we have a situation where rods moving to the right contract, and rods moving to the left expand. But this would provide a way of detecting absolute motion, and is forbidden by the first postulate. The only solution is for lengths perpendicular to the direction of motion to remain unchanged. (A similar argument for lengths parallel to the direction of motion might be used in an attempt to disprove length contraction. It would fail because marks made simultaneously at different x-coordinates in S appear to be made at different times in S'.)

3.7 Derivation of the Lorentz transformation

A more systematic way of obtaining consequences of Einstein's postulates is to find the transformation equations between the coordinates of the same event in the inertial systems S and S'. The Gallilean transformation is obviously incorrect as it does not allow for the constant velocity of light, so we shall now proceed to deduce the correct relativistic transformation, named after Lorentz. Since measurements of time in S and S' differ, all four coordinates describing an event (x, y, z **and** t) must be included in the transformation equations, whereas previously the time t was omitted, because classically it is the same in all coordinate systems.

A particle with constant velocity in S also moves with constant velocity in S', so the transformation equations must be linear. We have seen that lengths perpendicular to the direction of v are unchanged, so we have $y' = y, z' = z$ as before. The most general form of linear transformation for the remaining coordinates is

$$x' = Ax + Bt, t' = Cx + Dt,$$

where the constants A, B, C and D are functions of v and of the velocity of light c. To obtain exact expressions for these

constants we shall use two direct consequences of the postulates: the velocity of light must be c in both S and S', and relative motion must be described correctly, i.e. the origin O must appear to move backwards in S' with velocity v.

Suppose that light signals are sent out from O' in all directions at time $t' = 0$ (i.e. when O and O' coincide). Since they travel with velocity c, the envelope of all points reached by the signals is

$$x'^2 + y'^2 + z'^2 - c^2 t'^2 = 0,$$

or, substituting for x', y', z' and t'

$$(Ax + Bt)^2 + y^2 + z^2 - c^2(Cx + Dt)^2 = 0. \tag{3.1}$$

In S, however, the signals also appear to be sent out from the origin at time $t = 0$ and to travel with velocity c, so that their envelope should be simply

$$x^2 + y^2 + z^2 - c^2 t^2 = 0. \tag{3.2}$$

Equating coefficients in (3.1) and (3.2) gives three equations

$$[x^2] \quad A^2 - c^2 C^2 = 1, \tag{3.3}$$

$$[xt] \quad 2(AB - c^2 CD) = 0, \tag{3.4}$$

$$[t^2] \quad B^2 - c^2 D^2 = -c^2. \tag{3.5}$$

A fourth comes from the expected velocity of O in S', which is

$$\frac{x'}{t'} = \frac{A0 + Bt}{C0 + Dt} = \frac{B}{D} = -v. \tag{3.6}$$

Combining (3.5) and (3.6)

$$D = \frac{1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}, \quad B = \frac{v}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}},$$

and substituting into (3.3) and (3.4) gives

$$A = \frac{1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}, \quad C = \frac{-v^2/c^2}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}.$$

We usually write γ for $\dfrac{1}{\sqrt{\left(1-\dfrac{v^2}{c^2}\right)}}$.

The full transformation is then

$$x' = \gamma\,(x - vt),\ y' = y,\ z' = z,\ t' = \gamma\left(t - \frac{xv}{c^2}\right).$$

3.8 Properties of the Lorentz transformation

Although the Lorentz transformation is not as simple as the Gallilean, it reduces to the latter when $v \ll c$. (Then $\gamma \to 1$, $v/c^2 \to 0$.) Reduction to the classical case in the limit of small velocities will turn out to be a general feature of relativistic formulae, as we might expect since this limit reproduces exactly those conditions under which classical physics was so successful for so long. The inverse Lorentz transformation, giving (x, y, z, t) in terms of (x', y', z', t'), may be obtained by direct solution as

$$x = \gamma\,(x' + vt'),\ y = y',\ z = z',\ t = \gamma\left(t' + \frac{x'v}{c^2}\right).$$

It can also be obtained by exchanging primed and unprimed quantities in the direct transformation and changing the sign of v, as we expect because of the symmetry between coordinate systems implied in the first postulate.

The Lorentz transformation includes time dilation and length contraction. In the light pulse experiment the coordinates of the arrival of the pulse at P are $(vT, 0, 0, T)$ in S and $(0, 0, 0, T')$ in S'. Applying the transformation

$$T' = \gamma\left(T - \frac{v^2T}{c^2}\right) = T\sqrt{\left(1 - \frac{v^2}{c^2}\right)},$$

as obtained earlier. The length of a moving object is defined as the distance between the positions of its ends. Both measurements must be made at the same time, or the object will have moved in the interval, and the result will be meaningless. Consider measurements made in S' at time $t' = 0$ to find the length of a rod which is at rest in S with its ends at $x = 0$ and $x = L$. Using the transformation equation

$$x = \gamma\,(x' + vt'),$$

the positions of the ends are $x' = 0$ and

$$x' = \frac{L}{\gamma},$$

confirming that a moving rod appears to be contracted. (**Exercise.** Show that, in S, one measurement takes place at $t = 0$ and the other at $t = Lv/c^2$.)

3.9 The symmetry between coordinate systems

Fig. 11 shows a version of the light pulse experiment in which the photocell P is placed at the origin O and the reflection from the mirror is perpendicular in S. In S', the light pulse, O, and

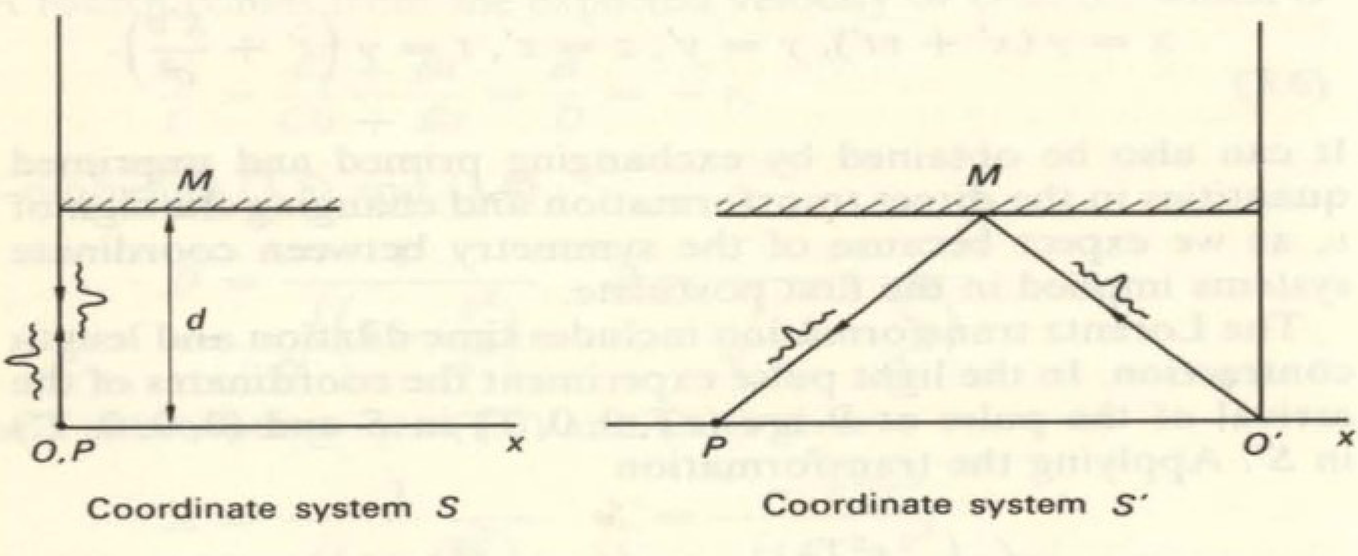


Fig. 11

P all travel backwards. The observer at O now measures the time taken by the pulse as a proper time and so we expect to find

$$T = T' \sqrt{\left(1 - \frac{v^2}{c^2}\right)}.$$

(**Exercise.** Repeat the previous analysis to confirm this.) This illustrates an important property of time dilation and length contraction: both effects are completely symmetrical between coordinate systems. For example, to measure in S the length of a rod at rest in S' we note the positions of its ends at the same time in S. But in S' these measurements appear to be made at different times and so a correct value for the length is not to be expected.

Finally, consider the hypothetical light signals used in deriving the Lorentz transformation. At time T' in S' the signals have reached points on the surface of a sphere of radius cT'. This surface does not appear to be spherical in coordinate system S but then in this coordinate system the points on it correspond to events which happen at different times. If we do consider the points reached by the light signals after equal times T in S, they do indeed lie on the surface of a sphere of radius cT, as expected.

3.10 The twin paradox

Suppose one of a pair of identical twins takes a quick return trip by space-ship to a nearby star, while the other stays behind on earth. The earth twin expects the astronaut twin's clocks to run slow throughout his journey, because of time dilation. Since time dilation affects all possible kinds of clock, including biological processes such as pulse or heartbeats, he therefore concludes that the astronaut twin ages more slowly and will be the younger of the two when they meet again. What does the astronaut twin think? Since he sees the earth twin move away and then approach again, our first impulse is to conclude that the experiences of the two twins are quite symmetrical. But in that case the astronaut twin will expect the earth twin to be the younger when they meet.

There cannot be two correct answers. The twins must be able to agree unambiguously on which of them is the younger when they meet. Fortunately, if we look carefully at the situation, we find that there is no paradox after all. The experiences of the two twins are *not* symmetrical: the astronaut twin feels an acceleration when he turns round to come back to earth, while the earth twin observes nothing of the kind. In the next chapter we shall look at what happens to each twin in more detail, and conclude that they will agree on their respective ages when they meet again. For some reason this 'paradox', although easily and unambiguously resolvable using Special Relativity alone, still crops up from time to time in the literature. Perhaps in a few years time, when the answer can be demonstrated experimentally, it will be finally laid to rest.

Additional exercises

1. The nearest star (Alpha Centauri) is 4 light years from earth. How fast must a space-ship travel to get there in 2 years, as measured by the occupants of the space-ship? How far will the occupants think they have travelled when they arrive at the star? Is your result in agreement with the length contraction formula?

(*Answer*. $\sqrt{(4/5)}c$; $4/\sqrt{5}$ light years)

2. A space-ship flying over the surface of a planet at velocity $0{\cdot}8c$ simultaneously lowers two probes 10 m apart. How far apart are the points where the probes strike the surface? Will an inhabitant of the planet see both probes strike the surface simultaneously?

(*Answer*. 6 m; No. There is a time difference of $4{\cdot}4 \times 10^{-8}$ s)

3. A space-ship (S') travelling at velocity $0{\cdot}8c$ passes earth (S) at time $t = t' = 0$. One hour later (by ship's clocks) a radio signal is sent back to earth. What are the times of sending and receiving the signal (a) in S, (b) in S'?)

(*Answer*. (a) sent after 1 h 40 min, received after 3 h; (b) sent after 1 h, received after 5 h)

4. Show that the proper time interval between two events corresponding to the emission and reception of a light pulse is always zero.

5. If event A causes event B, the time interval between A and B must be less then the time taken for light to go from A to B (or no signal can be sent to make B happen). If this is the case, show that a coordinate system always exists in which events A and B are simultaneous.

4

Velocities and accelerations

4.1 Transformation of velocities in one dimension

What is the apparent velocity u', as seen in S', of a particle with velocity u in coordinate system S? If both velocities are parallel to the transformation velocity v, by definition $u = \frac{dx}{dt}$, $u' = \frac{dx'}{dt'}$. (The general case will be considered later.) Straightforward differentiation of the Lorentz transformation equations gives

$$dx' = \gamma\,(dx - v\,dt) \text{ and } dt' = \gamma\left(dt - \frac{v}{c^2}dx\right),$$

and leads to

$$u' = \frac{dx'}{dt'} = \frac{\gamma(dx - v\,dt)}{\gamma\left(dt - \frac{v}{c^2}dx\right)} = \frac{u - v}{1 - \frac{uv}{c^2}}. \tag{4.1}$$

As we have seen, the Gallilean transformation gives

$$u' = u - v.$$

Again the relativistic result reduces to the classical one for $v \ll c$. The inverse transformation may be obtained by direct solution or by exchanging primed and unprimed quantities and changing the sign of v. Both methods give

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}. \tag{4.2}$$

The constancy of the velocity of light is built into this transformation law. We can obtain the velocity of light in system S from the value in S' by putting $u' = c$ in equation (4.2). The answer comes out to be $u = c$, as expected. For material objects, c represents a limiting velocity which can be approached but never surpassed. As an example, suppose a space-ship passing by the earth at velocity $0{\cdot}9c$ fires a rocket straight ahead with velocity $0{\cdot}9c$ relative to the ship. Using equation (4.2), the velocity of the rocket with respect to the earth is not $1{\cdot}8c$, but only

$$\frac{0{\cdot}9 + 0{\cdot}9}{1 + (0{\cdot}9)^2}\,c = \frac{1{\cdot}80}{1{\cdot}81}\,c \approx 0{\cdot}9945c.$$

Later on we shall derive a formula for the variation of velocity with energy and show how it has been verified to a high degree of accuracy.

4.2 The exploding space-ship

This is a more complicated example of velocity transformations, but still only involves one dimension. Suppose a space-ship travelling at $0{\cdot}5c$ relative to the earth explodes and breaks into two equal halves. One half continues in the same direction as the original ship while the other comes backwards with velocity $0{\cdot}5c$ relative to earth. What is the velocity of the forward half?

Fig. 12(a) shows the situations before and after the explosion as seen in the earth coordinate system (S), and Fig. 12(b) the

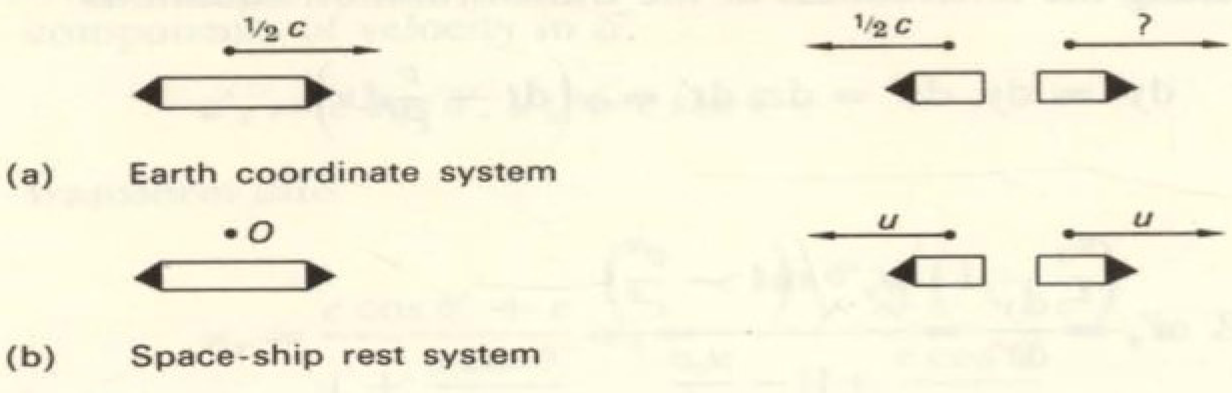


Fig. 12

same situations as they appear in an inertial coordinate system (S') moving with the original space-ship. (This is usually called the space-ship's 'rest-system'.) The transformation velocity v between S and S' is $0{\cdot}5c$, counting velocities in the forward direction as positive.

To conserve momentum in S', the two halves must have equal and opposite velocities u and $-u$. The velocity of the backward half transforms into system S as

$$\frac{-u + 0{\cdot}5c}{1 - 0{\cdot}5\,u/c},$$

but this is also equal to $-0{\cdot}5c$, so $u = 0{\cdot}8c$. Putting this value into the transformation for the forward half of the space-ship, its velocity relative to the earth comes out as

$$\frac{0{\cdot}8 + 0{\cdot}5}{1 + 0{\cdot}8 \times 0{\cdot}5}\,c = \frac{13}{14}\,c \approx 0{\cdot}93c.$$

4.3 The general transformation of velocities

Let a velocity $\mathbf{u} = (u_x, u_y, u_z)$ in S transform into $\mathbf{u}'$ (u'_x, u'_y, u'_z) in S'. The relation between u_x and u'_x must be

$$u_x{}' = \frac{u_x - v}{1 - \dfrac{u_x v}{c^2}}$$

from the one-dimensional result. For the other components, combining the differentials of the transformation equations

$$\mathrm{d}y' = \mathrm{d}y,\ \mathrm{d}z' = \mathrm{d}z,\ \mathrm{d}t' = \gamma\left(\mathrm{d}t - \frac{v}{c^2}\mathrm{d}x\right)$$

gives

$$u'_y = \frac{\mathrm{d}y'}{\mathrm{d}t'} = \frac{u_y\sqrt{\left(1 - \dfrac{v^2}{c^2}\right)}}{1 - \dfrac{u_x v}{c^2}},$$

$$u'_z = \frac{dz'}{dt'} = \frac{u_z\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}{1 - \frac{u_x v}{c^2}}.$$

We see that, unlike lengths, velocities perpendicular to the x-axis do not remain invariant. (Except for $v \ll c$, when the Gallilean transformation result is re-obtained.) The factor of $\sqrt{(1 - v^2/c^2)}$ in the numerator is present even if $u_x = 0$. It is a direct consequence of time dilation.

4.4 Transformations of angles

Angles change with the coordinate system even in classical physics. For example, rain falling vertically strikes a cyclist in the face at an angle which depends on his velocity. The relativistic effects are very similar, except that the correct transformation of velocities must be used. (**Exercise.** Suppose rain falls vertically with velocity u on a cyclist riding along a flat road with velocity v. Show that he observes rain falling at an angle $\tan^{-1}(\gamma v/u)$ to the vertical. How does this compare with the classical value? Would the average cyclist notice the difference?)

A more important effect occurs in the emission of light from moving atoms. Consider the light emitted at an angle θ' to the direction of motion in the rest-system (S') of an atom moving with velocity v in the laboratory (coordinate system S). Its components of velocity in S'

$$u'_x = c\cos\theta', u'_y = c\sin\theta'$$

transform into

$$u_x = \frac{c\cos\theta' + v}{1 + \frac{v\cos\theta'}{c}}, u_y = \frac{c\sin\theta'\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}{1 + \frac{v\cos\theta'}{c}}, \text{ in } S.$$

But u_y must be equal to $c \sin \theta$, where θ is the angle to the x-axis in S, so we have obtained a relation between the two angles of the form

$$\sin \theta = \frac{\sin \theta' \sqrt{\left(1 - \frac{v^2}{c^2}\right)}}{1 + \frac{v \cos \theta'}{c}}. \tag{4.3}$$

(**Exercise.** Use u_x to obtain a similar expression for $\cos \theta$. Check that $\cos^2\theta + \sin^2\theta = 1$.)

Many atoms emit light isotropically, i.e. uniformly in all directions in their rest-system, in which case half of the light goes forward of $\theta' = 90°$. But equation (4.3) shows that this angle corresponds in the laboratory to $\sin \theta = \sqrt{(1 - v^2/c^2)}$, which can be very small if the atom is moving with a velocity approaching that of light. In such a case, almost all the light is concentrated into a small cone about the direction of motion of the atom, as illustrated in Fig. 13.

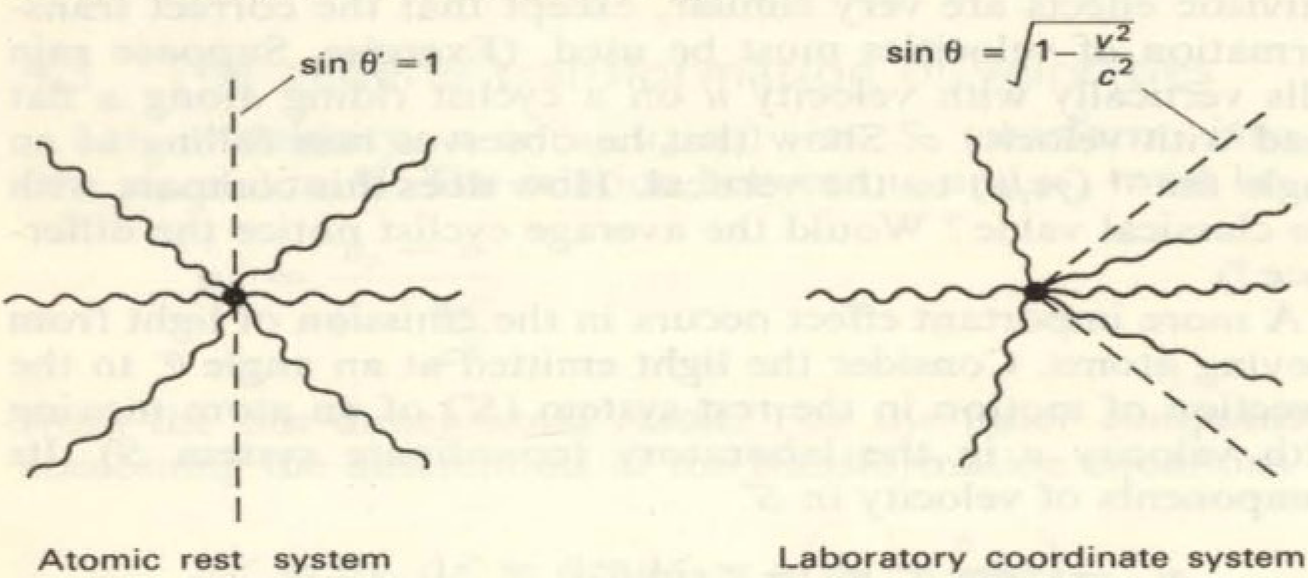


Fig. 13

4.5 The Doppler effect

Is the frequency of light emitted by a moving atom changed, as well as its direction? Classical physics predicts such a change, called the Doppler effect. (An example of the Doppler effect

for sound waves is the way in which the pitch of a train whistle or car horn drops as the vehicle goes past.) Classically, the ratio of the observed to the emitted frequency is

$$\frac{\nu_1}{\nu_0} = \frac{c+v}{c}$$

if the source of the waves is moving along the line of sight with velocity v, or

$$\frac{\nu_1}{\nu_0} = \frac{c-v}{c}$$

if the source is fixed and the observer moves. Both formulae reduce to

$$\frac{\nu_1 - \nu_0}{\nu_0} \approx \frac{v}{c}$$

if $v \ll c$.

This existence of two different formulae, depending on whether the source or the observer is moving, contrasts with the essential sysmmetry between coordinate systems implied in Einstein's first postulate. Relativistically, we expect to find a single formula, depending only on the relative velocity of source and observer. To derive it we shall use the concept of an invariant, i.e. a quantity which is the same in all inertial coordinate systems.

The invariant needed in this case is the phase of the electromagnetic wave. Writing the one-dimensional wave amplitude as

$$y = A \sin\left\{2\pi\nu\left(\frac{x}{c} - t\right)\right\}$$

the phase ϕ is the quantity

$$2\pi\nu\left(\frac{x}{c} - t\right).$$

Its value for a particular pair of values (x, t) determines whether the wave has a crest $(\sin\phi = +1)$ or a trough $(\sin\phi = -1)$,

or is in some intermediate state at that point of space and time. Two observers in different inertial coordinate systems who examine the state of the wave at one place and time (i.e. at points (x, t) and (x', t') related by the Lorentz transformation) must obviously agree on the presence or absence of a crest or trough there. This is equivalent to saying that the phase of the wave is the same in both the coordinate systems, i.e. that the phase is invariant.

We shall now find the frequency ν in the laboratory (coordinate system S) of light emitted with frequency ν' in the rest-system (S') of an atom moving with velocity v. (This arrangement of coordinate systems is the same as in the last section.) From the invariance of the phase we have

$$\nu'\left(\frac{x'}{c} - t'\right) = \nu\left(\frac{x}{c} - t\right),$$

and from the inverse Lorentz transformation

$$x = \gamma(x' + vt'), \quad t = \gamma\left(t' + \frac{x'v}{c^2}\right).$$

Substituting we have

$$\nu'\left(\frac{x'}{c} - t'\right) = \nu\gamma\left(\frac{x' + vt'}{c} - t' - \frac{x'v}{c^2}\right)$$

$$= \nu\gamma\left(1 - \frac{v}{c}\right)\left(\frac{x'}{c} - t'\right).$$

If we now identify ν with ν_1 and ν' with ν_0, the result is

$$\frac{\nu_1}{\nu_0} = \frac{\nu}{\nu'} = \frac{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}{1 - \frac{v}{c}} = \sqrt{\left(\frac{c + v}{c - v}\right)}, \tag{4.4}$$

which, as expected, involves only the relative velocity v of source and observer. For $v \ll c$, this formula also reduces to

$$\frac{\nu_1 - \nu_0}{\nu_0} \approx \frac{v}{c}.$$

To allow for source or observer motions which are not directly along the line of sight, equation (4.4) must be modified. Applying a similar invariance argument in two dimensions shows that the factor $(1 - v/c)$ in the denominator should be replaced by $(1 - v\cos\theta/c)$, where θ is the angle between the relative velocity and the line of sight, but the factor $\sqrt{(1 - v^2/c^2)}$ in the numerator (which follows directly from time dilation) should remain unchanged. The final result is therefore

$$\frac{\nu_1}{\nu_0} = \frac{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}{1 - \frac{v\cos\theta}{c}}. \tag{4.5}$$

A particularly interesting case occurs for $\theta = 90°$, i.e. for motion exactly transverse to the line of sight. Although no effect is predicted on the classical theory, a relativistic one should exist because of time dilation. The effect is only proportional to $\frac{v^2}{c^2}$, and unless the apparatus is set up very carefully may well be hidden by the ordinary Doppler effect owing to a very small component of velocity along the line of sight. Ives and Stilwell (1938) did, however, succeed in detecting this transverse Doppler effect in the light emitted from rapidly moving atoms.

4.6 A resolution of the twin paradox

We are now in a position to consider the experiences of each twin in more detail. To avoid any difficulties with accelerations we will assume that the astronaut twin makes his journey in a way which involves only Special Relativity, by travelling outwards in a space-ship moving with constant velocity v, and transferring to a second ship moving with equal velocity in the opposite direction for his return journey. We will also assume that each twin carries a light which flashes every time his heart beats, and that each twin's heart beats steadily once a second in his rest-system.

Suppose the earth twin measures the total time of the trip

as T seconds on his own clock, and calculates how much time has elapsed for the astronaut twin by counting the number of light flashes he receives. Using equation (4.4) to allow for the Doppler effect, $\sqrt{\left(\frac{c-v}{c+v}\right)}$ flashes arrive at earth per second during the outward journey and $\sqrt{\left(\frac{c+v}{c-v}\right)}$ arrive per second during the return. The change from one rate to the other occurs after $\frac{1}{2}T + \frac{1}{2}vT/c$ seconds, because the astronaut twin changes ships a distance $\frac{1}{2}vT$ from the earth, and we must allow for the time of travel of the light. The total number of flashes counted by the earth twin is therefore

$$\frac{T}{2}\left(1+\frac{v}{c}\right)\sqrt{\left(\frac{c-v}{c+v}\right)}+\frac{T}{2}\left(1-\frac{v}{c}\right)\sqrt{\left(\frac{c+v}{c-v}\right)} = T\sqrt{\left(1-\frac{v^2}{c^2}\right)},$$

and he concludes that the astronaut twin, having lived through fewer heartbeats than himself, will have aged less by the time he returns. This answer agrees with the ordinary time dilation formula.

By the astronaut twin's clock, the whole trip takes $T\sqrt{(1 - v^2/c^2)}$ seconds, equally divided between outward and return journeys. Because of the Doppler effect, he receives $\sqrt{\left(\frac{c-v}{c+v}\right)}$ light flashes per second from the earth twin on his way out and $\sqrt{\left(\frac{c+v}{c-v}\right)}$ per second on the way back. The total number of flashes he receives is

$$\frac{T}{2}\sqrt{\left(1-\frac{v^2}{c^2}\right)}\left\{\sqrt{\left(\frac{c-v}{c+v}\right)}+\sqrt{\left(\frac{c+v}{c-v}\right)}\right\} = T,$$

and so he comes to exactly the same conclusion as the earth twin – space travel is a good way of staying young! A careful look at the problem has disposed of the paradox, at least for this way of making the journey. The more usual case, involving accelerations, is considered in Chapter 8.

4.7 Accelerated motion

A space-ship travels along Ox at a constant acceleration a_0, as measured by accelerometers aboard. How far does it get in a given time?

Let the space-ship have velocity v at time t, and suppose that S' is the inertial coordinate system in which the space-ship is instantaneously at rest. After a short time $\Delta t'$ (which will become infinitesimal in the limit), the velocity of the space-ship in S' will be

$$\Delta v' = a_0 \Delta t'.$$

Transforming this velocity into system S (using the relativistic formula, since v may be comparable with c) produces

$$\frac{v + \Delta v'}{1 + \dfrac{v\Delta v'}{c^2}} = v + \Delta v,$$

where Δv is the apparent increase in velocity in system S. Expanding binomially, since $\Delta v' \ll c$, and ignoring terms in $(\Delta v')^2$

$$\Delta v = (v + \Delta v')\left(1 - \frac{v\Delta v'}{c^2}\right) - v = \Delta v' \left(1 - \frac{v^2}{c^2}\right).$$

But because of time dilation

$$\Delta t' = \Delta t\sqrt{\left(1 - \frac{v^2}{c^2}\right)},$$

so the apparent acceleration in S is

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \lim_{\Delta t' \to 0} \frac{\Delta v'}{\Delta t'}\left(1 - \frac{v^2}{c^2}\right)^{3/2}$$

$$= a_0 \left(1 - \frac{v^2}{c^2}\right)^{3/2}.$$

One integration of equation (4.6) gives

$$a_0 t = \frac{v}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} + \text{constant},$$

or, if $v = 0$ at $t = 0$,

$$v = \frac{a_0 t}{\sqrt{\left(1 + \frac{a_0{}^2 t^2}{c^2}\right)}} = \frac{c}{\sqrt{\left(1 + \frac{c^2}{a_0{}^2 t^2}\right)}}.$$

(The first form is useful at small t; the second as v approaches c.) A final integration gives the distance travelled as

$$x = \frac{c^2}{a_0}\sqrt{\left(1 + \frac{a_0{}^2 t^2}{c^2}\right)} - \frac{c^2}{a_0},$$

if $x = 0$ when $t = 0$.

We have managed to obtain this result using Special Relativity because only inertial coordinate systems have been used in our calculations. Any attempt to use non-inertial systems, such as asking about the experiences of the ship's passengers, does need General Relativity however, as we shall see in Chapter 8.

Additional exercises

1. Two space-ships have equal and opposite velocities of $0{\cdot}8c$ seen from earth. What is their relative velocity (a) as seen from earth, (b) as seen from one of the space-ships?

(*Answer*. (a) $\frac{16}{10}c$, (b) $\frac{40}{41}c$)

2. A particle travelling at a velocity of $0{\cdot}9c$ in the laboratory breaks up into two particles each with velocity $0{\cdot}85c$ in the particle's rest-system. What are their greatest and least velocities in the laboratory?

(*Answer*. $0{\cdot}991c$, $0{\cdot}213c$)

3. What is the apparent angle to Ox' in coordinate system S' of the following in S:

(a) a stationary rod inclined at 45° to Ox,
(b) the trajectory of a particle with velocity u at 45° to Ox,
(c) the trajectory of a light pulse travelling at 45° to Ox?

(*Answer*. (a) $\tan^{-1}(\gamma)$, (b) $\tan^{-1}\gamma^{-1}(1 - \sqrt{2}v/u)^{-1}$, (c) $\tan^{-1}\gamma^{-1}(1 - \sqrt{2}v/c)^{-1}$. $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$

4. A spectral line of rest wave-length 5×10^{-7} m in the light emitted by two distant galaxies in opposite directions as observed from earth, is received there with apparent wave-lengths of $7{\cdot}07 \times 10^{-7}$ m and $8{\cdot}66 \times 10^{-7}$ m. Calculate the apparent wave-length of the same line in light emitted by one of the galaxies and received by the other.

(*Answer*. $12{\cdot}25 \times 10^{-7}$ m)

5. In the version of the Twin Paradox described in this chapter, suppose that the astronaut twin travels out at speed v and returns at $\frac{1}{2}v$. Show that the ages of the twins are less different than in the previous case when they meet again, but that both twins still agree on the size of the difference.

6. A space-ship leaves earth and accelerates steadily at g ($9{\cdot}8$ m/s^2), as measured by its own accelerometers, until it reaches Alpha Centauri, 4 light years away. What is the length of the journey, as measured by earth clocks?

(*Answer*. 4·9 years)

5

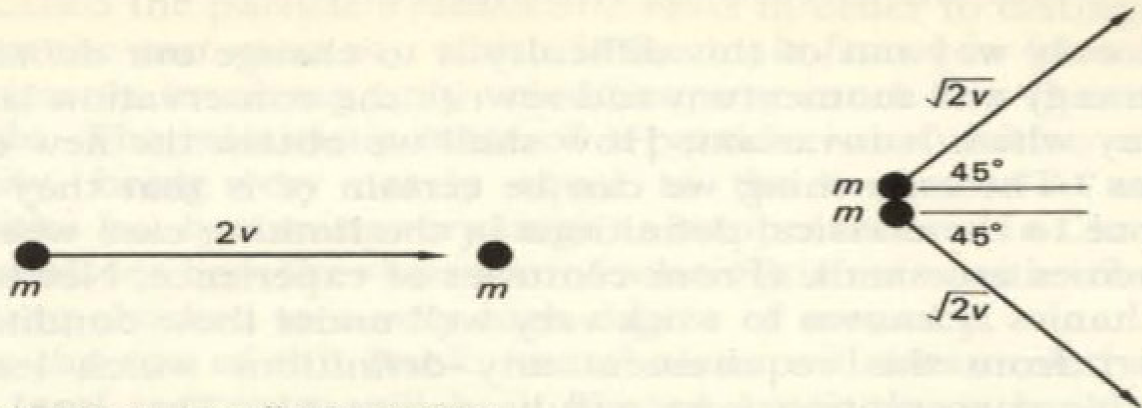
Relativistic mechanics I

5.1 The inadequacy of Newton's Laws

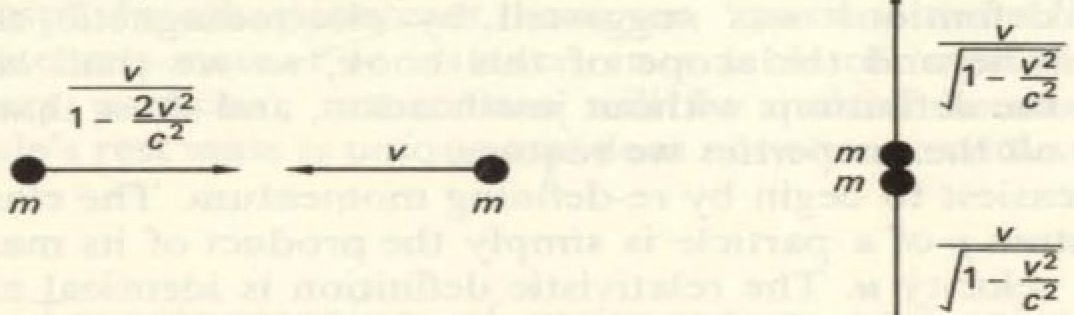
We saw in Chapter 2 that Newton's laws of motion are invariant under a Gallilean transformation. It follows that, classically, energy and momentum should be conserved in all inertial coordinate systems, and this was demonstrated, for a particular elastic collision of two billiard balls, in Fig. 5. Relativistically, we know it is correct to use the Lorentz transformation. What effect will this have on the conservation laws? A simple way of seeing the answer is to repeat our former example, using the classical definitions of energy and momentum together with the relativistic transformation law for velocities. The result is shown in Fig. 14.

Fig. 14 (a) is identical with Fig. 5(a), since the same conservation laws of energy and momentum have been applied in each. Fig. 14(b) shows the situations before and after the collision as they now appear in an inertial coordinate system moving with velocity v in the same direction as the incident ball. (This is the same coordinate system as in Fig. 5(b), although we shall see later that relativistically the total momentum in this system is no longer zero.) The effects of using the relativistic transformation of velocities are easily apparent: the incident ball has velocity

$$\frac{v}{1 - \dfrac{2v^2}{c^2}}$$



(a) Laboratory coordinate system



(b) Classical centre-of-mass coordinate system

Fig. 14

rather than v, and while both balls still end up travelling at right angles to the incident direction, their velocity is

$$\frac{v}{\sqrt{\left(1-\dfrac{v^2}{c^2}\right)}}$$

instead of v. Writing down the usual conservation equations for momentum (mass × velocity) and kinetic energy ($\frac{1}{2}$ mass × velocity^2) gives

(before collision) (after collision)

$$\text{momentum: } mv\left\{\left(1-\frac{2v^2}{c^2}\right)^{-1}-1\right\} \neq 0+0,$$

$$\text{energy: } \tfrac{1}{2}mv^2\left\{\left(1-\frac{2v^2}{c^2}\right)^{-2}+1\right\} \neq \tfrac{1}{2}mv^2\left\{2\left(1-\frac{v^2}{c^2}\right)^{-1}\right\}.$$

These equations do not balance, and we are forced to conclude that the conservation laws in their usual form are not invariant under Lorentz transformations.

5.2 The definition of momentum

The only way out of this difficulty is to change our definitions of energy and momentum and rewrite the conservation laws in a way which is invariant. How shall we obtain the new definitions? The only thing we can be certain of is that they must reduce to the classical definitions in the limiting case where all velocities are small. (From centuries of experience, Newtonian mechanics is known to work very well under these conditions.) Apart from this requirement any definitions which lead to sensible conservation laws will be acceptable. The usual form of the definitions was suggested by electromagnetic theory which is beyond the scope of this book, so we shall simply present the definitions without justification, and show that they possess all the properties we require.

It is easiest to begin by re-defining momentum. The classical momentum p of a particle is simply the product of its mass m_0 and its velocity u. The relativistic definition is identical except that it uses proper time, i.e. time as measured in the particle's rest-system. Putting in the time dilation factor, the formula is therefore

$$P = \frac{m_0 u}{\sqrt{\left(1 - \frac{u^2}{c^2}\right)}}.$$

This obviously reduces to the classical definition for $u \ll c$. (In fact we could use proper time in the classical definition; as there is no classical equivalent of time dilation this would not change the usual formula.)

5.3 Relativistic mass and rest mass

Because the classical and relativistic definitions of momentum are so similar we commonly write the latter as $p = mu$, where

$$m = \frac{m_0}{\sqrt{\left(1 - \frac{u^2}{c^2}\right)}}.$$

m is called the particle's *relativistic mass* in order to distinguish it from the *rest mass* m_0, which is the mass found in (classical) experiments involving only velocities very much less than that of light. The relativistic mass of a particle is a function of its velocity, being very nearly equal to the rest-mass at small velocities but becoming very large as u approaches c. In Chapter 7 we shall see how the variation of relativistic mass with velocity has been checked to a very high degree of accuracy.

For the rest of this book, 'mass' alone will mean relativistic mass, and rest mass will be indicated explicitly or by a zero subscript. In other contexts, however, 'mass' is more likely to indicate rest-mass. For instance, in a table of 'masses of elementary particles', rest-masses will be quoted because each particle's rest mass is unique and does not change with velocity.

5.4 Transformation of momentum and relativistic mass

What is the apparent momentum p', in S', of a particle with momentum p in coordinate system S? To answer this question we must combine the relativistic definition of momentum with the relativistic transformation law for velocities. If p is parallel to the velocity v in the Lorentz transformation, the velocity of the particle in S' is

$$u' = \frac{u - v}{1 - \dfrac{uv}{c^2}},$$

and substituting this into the definition of p' gives

$$p' = m_0 u' \left(1 - \frac{u'^2}{c^2}\right)^{-1/2}$$

$$= \frac{m_0(u - v)}{1 - \dfrac{uv}{c^2}} \left\{ 1 - \frac{\left(\dfrac{u - v}{c}\right)^2}{\left(1 - \dfrac{uv}{c^2}\right)^2} \right\}^{-1/2}$$

$$= m_0(u - v)\left\{\left(1 - \frac{uv}{c^2}\right)\right\}^2 - \left(\frac{u - v}{c}\right)^2\Bigg\}^{-1/2}$$

$$= m_0(u - v)\left\{1 - \frac{2uv}{c^2} + \frac{u^2v^2}{c^4} - \frac{u^2}{c^2} + \frac{2uv}{c^2} - \frac{v^2}{c^2}\right\}^{-1/2}$$

$$p' = m_0(u - v)\left\{\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{u^2}{c^2}\right)\right\}^{-1/2} = \gamma(p - vm),$$

which is the same as the Lorentz transformation law for x and t.

Extending this analogy with the Lorentz transformation we expect a general momentum $\mathbf{p} = (p_x, p_y, p_z)$ to transform as

$$p'_x = \gamma\,(p_x - vm), \quad p'_y = p_y, \quad p'_z = p_z.$$

If we use the definition of momentum and the general transformation law for velocities we do indeed get the same result, after much algebra. The inverse transformation

$$p_x = \gamma(p_x' + vm'), \quad p_y = p_y', \quad p_z = p_z',$$

can be obtained, as usual, by exchanging primed and unprimed quantities in the direct transformation and changing the sign of v.

The fourth Lorentz transformation equation suggests the mass transformation law

$$m' = \gamma\left(m - \frac{vp_x}{c^2}\right).$$

(**Exercise.** Obtain this result using the definition of m and the transformation law for velocities.) It may seem strange for the mass of a particle to vary from one coordinate system to another, but our study of the conservation laws will soon reveal the deep significance which lies behind this transformation law for relativistic mass.

5.5 The invariant form of the conservation laws

We shall now deduce the relativistic form of the conservation laws. It is convenient to use the rather abstract idea of a general collision process for this purpose, reserving application of the

laws in detail to particular problems for the next two chapters.

Conservation of momentum in coordinate system S for our general collision can be expressed by a vector equation of the sums of momenta on each side

$$\sum_i \mathbf{p}_i = \sum_j \mathbf{p}_j, \tag{5.1}$$

where the index i runs over all the incoming particles and j runs over all the outgoing ones. (Remember that this one equation is really three separate ones for x, y and z-components of momentum.) The law of conservation of momentum will be invariant if the existence of equation (5.1) in coordinate system S automatically implies the truth of the corresponding equation in system S', i.e.

$$\sum_i \mathbf{p}_i' = \sum_j \mathbf{p}_j'. \tag{5.2}$$

The y and z-component parts of the equation are obviously invariant because these momentum components are identical in S and S' for each particle separately. In the x-component part of equation (5.2) substitution from the transformation equations gives

$$\gamma \sum_i \mathbf{p}_{xi} - \gamma v \sum_i m_i = \gamma \sum_j \mathbf{p}_{xj} - \gamma\, v \sum_j m_j, \tag{5.3}$$

where the factors γ and γv have been taken outside the summations because they are the same for all particles. The meaning of equation (5.3) is that conservation of momentum will only be invariant if relativistic mass is conserved at the same time.

We can verify this conclusion by looking at the conservation of relativistic mass directly. The equation

$$\sum_i m_i = \sum_j m_j$$

in coordinate system S transforms into

$$\gamma \sum_i m_i - \frac{\gamma v}{c^2} \sum_i p_{xi} = \gamma \sum_j m_j - \frac{\gamma v}{c^2} \sum_j p_{xj}$$

in system S'. (Again i runs over incoming particles and j over outgoing ones.) Once more we see that the invariance of one conservation law depends on the existence of the other.

To summarise, if momentum and relativistic mass are *both* conserved in any one inertial coordinate system, then these conservation laws are invariant, i.e. they are also true in all inertial coordinate systems related to the first by Lorentz transformations. These conservation laws satisfy our requirements mathematically; next we must look at their physical content.

5.6 Total energy and $E = mc^2$

Classically, momentum, energy, and rest mass are all conserved in a collision. (The latter because particles cannot be created or destroyed in the collision.) Relativistically, only momentum and relativistic mass are conserved. This apparent discrepancy can be resolved by considering the form of the relativistic mass for $u \ll c$. If we multiply by c^2 and expand binomially, retaining only powers of u upto u^2, we have

$$E = mc^2 = \frac{m_0c^2}{\sqrt{\left(1 - \frac{u^2}{c^2}\right)}} \approx m_0c^2 + \tfrac{1}{2}m_0u^2. \tag{5.4}$$

The first term is just the rest mass ($\times\ c^2$) and the second the classical kinetic energy. We see that, in the classical limit, conservation of relativistic mass implies both conservation of rest-mass and conservation of kinetic energy. The quantity mc^2 is called the *total energy* and is usually denoted by E.

5.7 The equivalence of mass and energy

In classical physics, mass and energy are two quite different quantities: they obey separate conservation laws and there is no question of converting one into the other. Equation (5.4) shows that this is no longer true in relativistic mechanics, at least for kinetic energy. Rest mass and kinetic energy are no more than different names for the two components of the total energy. They are usually measured in different units, with a conversion factor c^2, but this is just an historical accident and does not

change the fact that mass and energy are really only two different manifestations of the same basic quantity. (In a later chapter we shall see how to choose a set of units so that even this apparent difference disappears.) Conversion of rest mass into kinetic energy happens all the time. Take for instance the muon decays used in Chapter 3 in a test of time dilation: the sum of the rest masses of the decay products (electron, neutrino, anti-neutrino) is much less than the rest mass of the muon; the difference is converted during the decay, at a rate of c^2 per unit mass, into the kinetic energy of motion of the decay products.

Equation (5.4) really only expresses mass-energy equivalence for kinetic energy. Einstein extended this to cover all kinds of energy, making the equation $E = mc^2$ not only one of the best-known equations in physics, but also one of the most important. Chemical energy, electric and magnetic energy, any kind of potential energy, all are convertible into mass at the standard rate. In an exothermic chemical reaction, the products must weigh less than the reactants and vice versa for an endothermic reaction. A charged condenser or a coiled spring gain in mass because of the stored energy. All these are consequences of mass-energy equivalence. Since the rate of conversion c^2 is such a large number, most of these effects are not detectable with present-day apparatus. For example a 2-volt accumulator, charged to provide 40 ampere-hours, contains $2 \times 40 \times 3600 = 280\,000$ Joules of energy, but only increases in mass by $(2{\cdot}8 \times 10^5)/(9 \times 10^{16}) = 3{\cdot}2 \times 10^{-12}$ kilogrammes. Even the most exothermic chemical reactions only convert a vanishingly small proportion of their mass into energy.

Mass-energy conversion on a larger scale takes place in nuclear reactions. The reactors in nuclear power stations rely on the breaking up of uranium nuclei into lighter products, about one-thousandth of the rest mass of the uranium being converted into energy in the process. Complete fission of all the nuclei in one kilogram of uranium should therefore produce one gram of heat, i.e. 9×10^{13} Joules or about 10 000 megawatt-hours. (Unfortunately, fission is only possible for a small proportion of the nuclei.) The use of nuclear reactions in experimental tests of mass-energy equivalence will be described in Chapter 7,

where we shall also see how, in the annihilation of positrons (anti-electrons), the whole of the rest mass of a particle may be converted into (electromagnetic) energy.

5.8 Summary

In the next chapter we shall discuss some typical problems involving relativistic mechanics. It will be convenient first to summarise all the results obtained so far in this chapter, and to obtain a few other useful formulae.

A particle with rest mass m_0 and velocity u has relativistic mas.

$$m = \frac{m_0}{\sqrt{\left(1 - \frac{u^2}{c^2}\right)}}$$

and momentum

$$p = mu.$$

Its rest energy is

$$m_0c^2,$$

and its total energy

$$E = mc^2 = \frac{m_0c^2}{\sqrt{\left(1 - \frac{u^2}{c^2}\right)}}.$$

The transformation equations for momentum and energy are

$$p'_x = \gamma\left(p_x - \frac{vE}{c^2}\right), \quad p'_y = p_y, \quad p'_z = p_z,$$

$$E' = \gamma(E - vp_x).$$

Two relations which follow directly are

$$E^2 = m_0{}^2c^4\left(1 - \frac{u^2}{c^2}\right)^{-1} = m_0{}^2c^2u^2\left(1 - \frac{u^2}{c^2}\right)^{-1} + m_0{}^2c^4 = p^2c^2 + m_0{}^2c^4, \quad (5.5)$$

and

$$\frac{pc}{E} = \frac{muc}{mc^2} = \frac{u}{c}. \tag{5.6}$$

The relativistic kinetic energy T is defined as the difference of total and rest energies, i.e.

$$T = E - m_0c^2 = (m - m_0)c^2.$$

Substituting for E in equation (5.5)

$$p^2c^2 = (T + m_0c^2)^2 - m_0{}^2c^4 = T^2 + 2Tm_0c^2. \tag{5.7}$$

When solving problems it is a good idea to work with total energy and momentum rather than kinetic energy and velocities as this usually simplifies the algebra.

Additional exercises

1. The total energy of a particle is twice its rest energy. What is the ratio of its momentum to its kinetic energy?

(*Answer*. $\sqrt{3}/c$)

2. A particle of rest mass m_0 travelling at a velocity of $0{\cdot}6c$ strikes a similar particle at rest and the two stick together. What are the velocity and rest mass of the composite particle?

(*Answer*. $\frac{1}{3}c$, $3m_0/\sqrt{2}$)

3. The intensity of radiation from the sun which falls on the earth is $1{\cdot}4\ \mathrm{kW/m^2}$. How much of the sun's mass must be converted into energy every second? (The distance of the earth from the sun is $1{\cdot}5 \times 10^{11}$ m.)

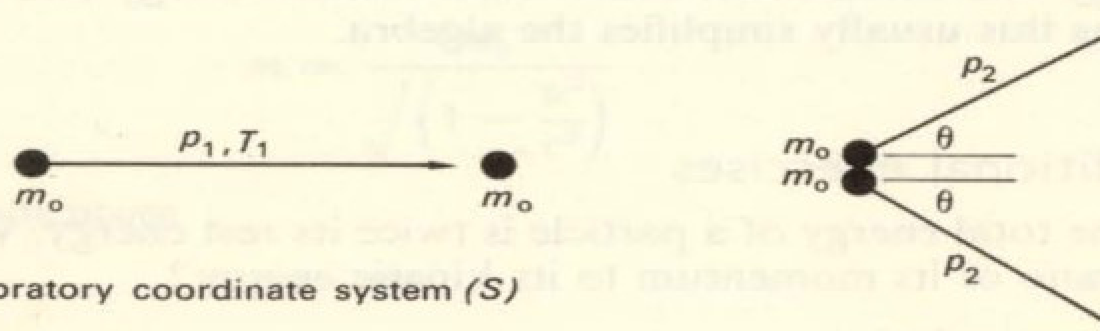
(*Answer*. $4{\cdot}4 \times 10^9$ kg)

6

Relativistic mechanics II

6.1 Examples of the use of the conservation laws

First, let us complete our analysis of the billiard ball collision which has already appeared in Figs. 5 and 14. Fig. 15(a) shows



(a) Laboratory coordinate system *(S)*

Fig. 15 (a)

the situations before and after the collision in the laboratory coordinate system (S). In this fully relativistic treatment the momentum of the incident ball is

$$p_1 = m_0 v \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

and its kinetic energy is

$$T_1 = m_0 c^2 \left[\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right].$$

It is easy to show that the angles θ in Fig 15(a) are not 45°, as in the classical case (see Fig. 5(a)). Calling the momentum of each ball after the collision p_2, the conservation equations are

momentum: $p_1 = 2p_2 \cos\theta,$ (6.1)

energy: $\sqrt{(p_1{}^2c^2 + m_0{}^2c^4)} + m_0c^2 = 2\sqrt{(p_2{}^2c^2 + m_0{}^2c^4)}.$ (6.2)

(We have used equation (5.5) to express total energies in terms of momenta and rest masses. Note that since we are equating *total* energies in equation (5.2) the rest energy m_0c^2 of the target ball must be included on the left-hand side.)
Substitution from (6.1) into (6.2) and squaring up gives

$$p_1{}^2c^2 + 2m_0{}^2c^4 + 2m_0c^2\sqrt{(p_1{}^2c^2 + m_0{}^2c^4)} = 4\,p_1{}^2c^2\sec^2\theta + 4m_0{}^2c^4. \quad (6.3)$$

Rewriting the square root as $T_1 + m_0c^2$ and cancelling terms in $m_0{}^2c^4$, we obtain

$$2m_0c^2T_1 = p_1{}^2c^2(\sec^2\theta - 1) = p_1{}^2c^2\tan^2\theta, \quad (6.4)$$

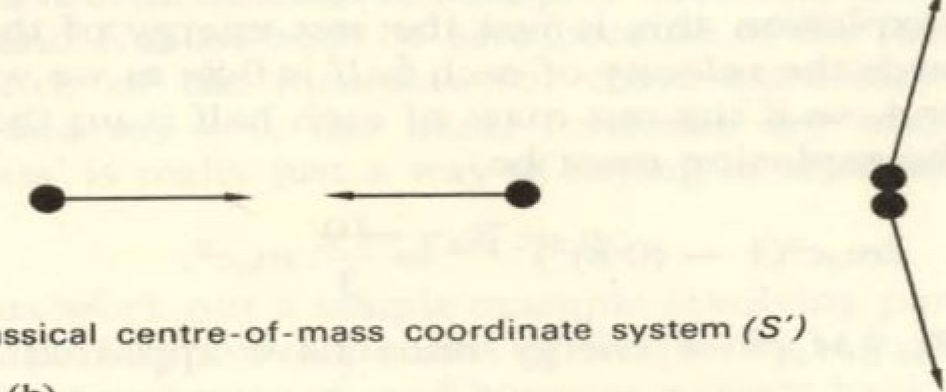
and finally, using equation (5.7) to express $p_1{}^2$ in terms of T_1

$$\tan^2\theta = \frac{2m_0c^2T_1}{2m_0c^2T_1 + T_1{}^2} \quad (6.5)$$

When T_1 is small and $T_1{}^2$ may be neglected, we re-obtain the classical result $\tan\theta = 1$. For larger values of T_1, θ is less than 45°.

This is a special case of a general result. Classically, if two equal mass particles collide elastically, the angle between them afterwards is always 90° in the laboratory coordinate system. Relativistically, the angle is less than 90°, by an amount depending on the velocity of the incident particle. For real billiard balls the effect is much too small to detect, but angles of less than 90° have been observed when fast α-particles (helium nuclei) are fired into a helium cloud chamber.

Fig. 15(b) shows the billiard ball collision as it appears in a coordinate system (S') moving with velocity v in the incident



(b) Classical centre-of-mass coordinate system (S')

Fig. 15 (b)

direction. Note that the two balls no longer come off at right angles to the incident direction or in opposite directions in this system. This is because S' is no longer the centre-of-mass system (i.e. the coordinate system in which the total momentum is zero). (**Exercise.** Show directly, or by calculating the total momentum p_{T} and energy E_{T} in the laboratory and transforming p_{T} to S', that the total momentum in the moving system is

$$m_0 v\left(1-\frac{v^2}{c^2}\right)^{-1/2}\left\{\left(1-\frac{4v^2}{c^2}\right)^{-1/2}-1\right\}.)$$

We can find the true centre-of-mass system by asking what Lorentz transformation of p_{T} and E_{T} is needed to make p'_{T} zero. Using the transformation equation for momentum

$$0=p'_{\mathrm{T}}=\left(1-\frac{w^2}{c^2}\right)^{-1/2}\left(p_{\mathrm{T}}-\frac{wE_{\mathrm{T}}}{c^2}\right),$$

we see that the appropriate transformation velocity is now

$$w=\frac{p_{\mathrm{T}}c^2}{E_{\mathrm{T}}}.$$

Any two particles must always have equal and opposite momenta in their centre-of-mass coordinate system, to make the total zero. This fact can be very useful in solving problems.

Finally, consider the exploding space-ship of Fig. 12. If the original ship has rest mass M_0, what is the rest mass of each of the halves into which it breaks up? The obvious answer $0{\cdot}5M_0$ is not correct, as we can see by considering conservation of energy in the rest system of the space-ship.

Before the explosion this is just the rest energy of the ship M_0c^2. Afterwards the velocity of each half is $0{\cdot}8c$ as we worked out in Chapter 4, so if the rest mass of each half is m_0 the total energy after the explosion must be

$$2m_0c^2(1-(0{\cdot}8)^2)^{-1/2}=\frac{10}{3}\;m_0c^2.$$

If $m_0=0{\cdot}5M_0$, $\frac{2}{3}M_0c^2$ of energy must have appeared from nowhere.

But energy is conserved and so we must have $m_0 = 0{\cdot}3M_0$. This means that the rest masses of the two halves which are finally projected apart make up only six-tenths of the rest mass of the original ship. The remainder is converted into energy and released in the explosion. (**Exercise.** Verify directly that the kinetic energy of each half after the explosion is $0{\cdot}2M_0c^2$.)

6.2 Particles with zero rest mass

Many experiments, including the photoelectric effect, are best explained by assuming that light is made up of particles called photons. The total energy of a photon is related to its frequency by

$$E = h\nu,$$

where h is Planck's constant, numerically equal to $6{\cdot}6 \times 10^{-34}$ J s. Obviously photons must travel at the velocity of light! Putting $u = c$ in equation (5.6) then shows that the momentum and energy of the photon must be related by

$$p = \frac{E}{c} = \frac{h\nu}{c}.$$

If $pc = E$ is substituted in equation (5.5) then

$$m_0 = 0,$$

and for this reason photons are often said to have zero rest mass. It is a favourite mistake to interpret 'zero rest mass' as meaning that p and E must both be zero because of the factor m_0 in the numerator of the formulae for these quantities. In fact for $u = c$ and $m_0 = 0$, the usual formulae are undefined. 'Zero rest mass' is really just a way of saying in shorthand

$$u = c,\ E = pc.$$

Let us work out a simple example involving particles of zero rest mass. Suppose a photon of frequency ν_1 collides with an electron of rest mass m_0 and bounces straight back, as in Fig. 16.

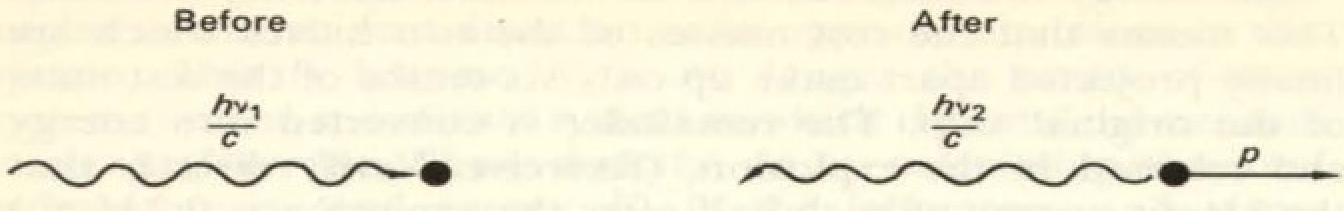


Fig. 16

What is the frequency ν_2 of the photon after the collision? Using the expression $h\nu/c$ for the photon momentum the conservation laws take the form

$$\text{momentum: } \frac{h\nu_1}{c} = p - \frac{h\nu_2}{c}, \tag{6.6}$$

$$\text{energy: } h\nu_1 + m_0c^2 = h\nu_2 + \sqrt{(p^2c^2 + m_0{}^2c^4)}, \tag{6.7}$$

where p is the momentum of the electron after the collision. Substituting for p from (6.6) into (6.7) and squaring up gives

$$2h\,(\nu_1 - \nu_2)\,m_0c^2 = 4h^2\nu_1\nu_2,$$

which enables ν_2 to be found in terms of ν_1, h and m_0. This is an example of the Compton effect, which can be observed with very short wavelength X-rays.

Another zero rest mass particle, the neutrino, is produced in the decays of some elementary particles. Fig. 17 illustrates the

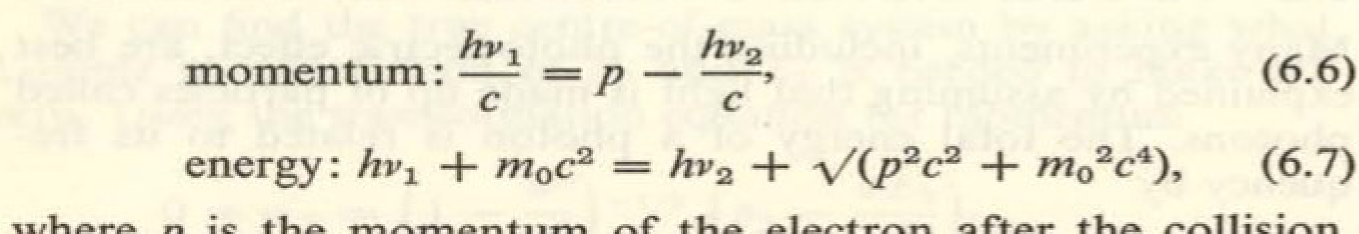


Fig. 17

decay of a particle of rest mass m_0 and velocity $0{\cdot}8c$ into a second particle of rest mass $m_0/3$ and a neutrino. The momentum and total energy of the original particle are $4m_0c/3$ and $5m_0c^2/3$, as measured in the laboratory. If the momenta of the decay

particle and the neutrino are p_1 and p_2 respectively, the conservation equations may be written as:

$$\text{momentum: } \tfrac{4}{3}m_0c = p_2 - p_1,$$

$$\text{energy: } \tfrac{5}{3}m_0c = \sqrt{(p_1{}^2c^2 + \tfrac{1}{9}m_0{}^2c^4)} + p_2c.$$

(Note that the energy of the neutrino is simply p_2c.) These two equations can be solved for p_1 and p_2. (**Exercise.** Show that $p_1 = 0$ i.e. the decay particle is at rest in the laboratory. What is the velocity of the neutrino?)

6.3 Invariant masses

It is easy to show directly from the Lorentz transformation that $x^2 - c^2t^2$ is an invariant. The proof goes as follows:

$$x'^2 - c^2t'^2 = \gamma^2\left\{(x - vt)^2 - c^2\left(t - \frac{xv}{c^2}\right)^2\right\}$$

$$= \gamma^2\left(1 - \frac{v^2}{c^2}\right)(x^2 - c^2t^2) = x^2 - c^2t^2.$$

We saw in the last chapter that p and m ($= E/c^2$) transform in the same way as x and t, so the quantity

$$E^2 - p^2c^2$$

must also be invariant. E and p may be the energy and momentum of a single particle (in which case the invariant is $m_0{}^2c^4$, where m_0 is the rest mass of the particle, by equation (5.5), or the total energy and momentum of a system of particles. In the latter case, the invariant may be written as

$$M^2c^4,$$

where M is a function of the rest masses of the particles in the system and also of their relative momenta. M is called the *invariant mass* of the system, and is always greater than the sum of the rest masses of all the individual particles. (**Exercise.** Show that the invariant mass of the two billiard balls in Fig. 15(a) is M, where $M^2c^4 = 2m_0c^2(T_1 + 2m_0c^2)$.)

Many of the shortest lived elementary particles (with lifetimes of 10^{-21} s or less) have only been discovered by the calculation of invariant masses. When these particles are produced in a cloud or bubble chamber, they decay before they have time to leave a visible track, but a computation of the invariant mass of the decay products will give an answer equal to the rest mass of the invisible particle. To find a new particle, the invariant mass of its possible decay products is calculated in a large number of cases. If these values vary widely a new particle is unlikely to exist, but occasionally all the values cluster around a particular number, which is interpreted as the rest mass of a very short-lived particle.

6.4 Forces

The usual way of defining forces in Special Relativity is by the equivalent of Newton's second law of motion. For the force acting on a particle with rest mass m_0 and velocity u this gives

$$F = \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{m_0 u\left(1 - \frac{u^2}{c^2}\right)^{-1/2}\right\}$$

$$= m_0\left(1 - \frac{u^2}{c^2}\right)^{-3/2}\frac{\mathrm{d}u}{\mathrm{d}t}.$$

Since the work done in moving the particle through a distance $\mathrm{d}x$ is

$$\mathrm{d}E = F\mathrm{d}x = m_0\left(1 - \frac{u^2}{c^2}\right)^{-3/2}\frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{d}x$$

$$= m_0\frac{\mathrm{d}x}{\mathrm{d}t}\left(1 - \frac{u^2}{c^2}\right)^{-3/2}\mathrm{d}u = m_0 u\left(1 - \frac{u^2}{c^2}\right)^{-3/2}\mathrm{d}u,$$

the total increase in energy in accelerating the particle from rest to velocity U is

$$\Delta E = \int_0^U m_0 u\left(1 - \frac{u^2}{c^2}\right)^{-3/2}\mathrm{d}u$$

$$= \left[m_0 c^2\left(1 - \frac{u^2}{c^2}\right)^{-1/2}\right]_0^U = (m - m_0)c^2,$$

or just the value we expect from the equivalence of mass and energy.

A detailed discussion of forces in Special Relativity is beyond the scope of this book. In the next chapter, however, we shall need an expression for the path of a particle of rest mass m_0 and charge e travelling at velocity u perpendicular to a magnetic field of strength H. Classically, equation of the magnetic and centrifugal forces gives the path as a circle of radius

$$r = \frac{m_0 u}{\mu_0 He}.$$

The relativistic result turns out to be identical with this except that the rest mass is replaced by the relativistic mass, making the formula

$$r = \frac{mu}{\mu_0 He} = \frac{m_0 u}{\mu_0 He}\left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}} = \frac{p}{\mu_0 He} \tag{6.8}$$

This expression is used to find the momentum of tracks in cloud and bubble chambers from their radii of curvature in a known magnetic field.

Additional exercises

1. A proton with rest mass m_0 and kinetic energy m_0c^2 collides with a stationary proton. What is the included angle between the protons after the collision? What is their invariant mass?

(*Answer*. $2\tan^{-1}(\sqrt{\frac{2}{3}})$ or 78·4°, $\sqrt{6}\, m_0$)

2. A particle of rest mass m_0 at rest in the laboratory decays into three particles of rest mass m_0. Two of these particles have velocities of $0{\cdot}8c$ and $0{\cdot}6c$ and directions at right-angles. Calculate the velocity and direction of the third particle, and the ratio of M_0 to m_0.

(*Answer*. $0{\cdot}836c$ at angle of $\tan^{-1}(9/16)$ to the direction opposite that of the first particle; 4·75)

3. A particle with rest mass m_0 and momentum p strikes an identical particle at rest. Find the transformation velocity to the centre-of-mass coordinate system and the momentum of each particle in the centre-of-mass.

(*Answer*. $pc^2/(\sqrt{(p^2c^2 + m_0{}^2c^4)} + m_0c^2)$, $\frac{1}{2}\sqrt{2m_0(\sqrt{(p^2c^2 + m_0{}^2c^4)} - m_0c^2)}$)

4. A photon of frequency ν rebounds straight back from a collision with a moving electron of rest mass m_0. The frequency of the photon is unchanged but the electron is brought to rest. What was the total energy of the electron?

(*Answer*. $\sqrt{(4h^2\nu^2 + m_0{}^2c^4)}$)

5. If the decay in Fig. 17 is reversed so that the decay particle now travels in the same direction as the original, what is its velocity in the laboratory?

(*Answer*. $40c/41$)

6. A 'photon rocket' fires radiation backwards as its propellant. If the initial rest mass of the rocket is m_0 and its final velocity v, show that its final rest mass is $m_0\sqrt{(c - v)/(c + v)}$.

7. A proton and an α-particle (He^4 nucleus) have equal velocities. What is the ratio of the radii of curvatures of their tracks in the same magnetic field?

(*Answer*. 1:2)

7

Experimental tests and consequences of Special Relativity

In the last four chapters we have deduced many consequences of Einstein's two postulates about observations in inertial coordinate systems. Although hypothetical experiments were used in many of these deductions, we have not always been able to quote real experimental evidence in confirmation. In contrast, the formulae of relativistic mechanics, as developed in the last two chapters, can be tested in many ways. These tests provide the strongest, albeit somewhat indirect, evidence that the original postulates are correct.

Relativistic effects are difficult to detect in experiments with 'classical' apparatus, where velocities are usually much smaller than c and the kinetic energy is only a very tiny fraction of the total energy. (For example, if a car weighing 500 kg travels at 200 (km/h), less than one part in 10^{13} of its total energy is kinetic.) So although some departures from Newton's laws should exist in billiard ball collisions, in practice they are too small to detect.

The situation is very different for sub-atomic particles, which can be accelerated to very high velocities because of their very small masses. As their velocity approaches that of light and their total energy becomes many times their rest mass, relativistic effects are demonstrated on a very large scale. This chapter will discuss ways in which sub-atomic particles have been used to verify the formulae of relativistic mechanics and other predictions of Special Relativity.

7.1 The variation of relativistic mass with velocity

Many experiments performed to check the relation

$$m = m_0 \left(1 - \frac{u^2}{c^2}\right)^{-1/2}$$

have used electrons as test particles. A combination of deflections in electric and magnetic fields is used to measure the charge-to-mass ratio e/m for the electron, and since relativity theory (though beyond the scope of this book) predicts that the charge e should be an invariant, this is quite as good as if m could be measured directly.

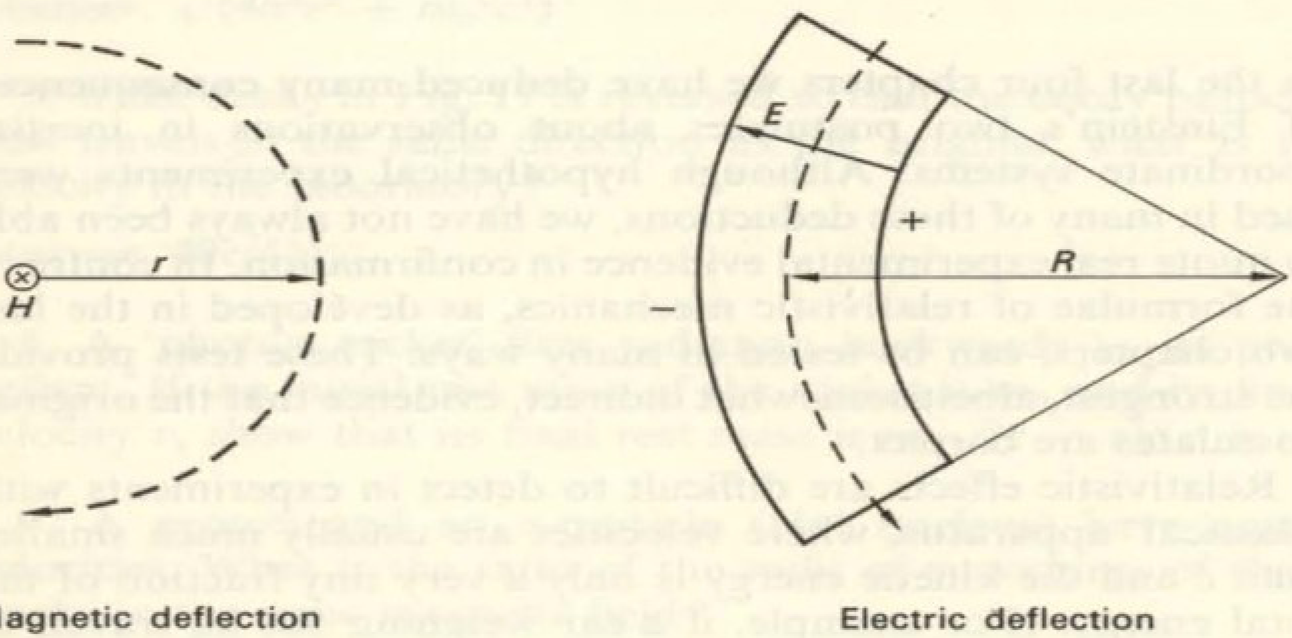


Fig. 18

Fig. 18 illustrates the principle of an experiment which was carried out by Rogers et al. in 1939, using electrons of sharply defined velocity from the radioactive decay of radium nuclei. The radius of curvature r of the electrons' path in a magnetic field of strength H was measured, together with the electric field E needed to deflect the electrons in an orbit of radius R in a specially constructed cylindrical condenser. Combining the equations

$$E = \frac{mu^2}{Re} \quad \text{and} \quad r = \frac{mu}{\mu_0 He}$$

gives

$$u = \frac{ER}{\mu_0 Hr} \quad \text{and} \quad \frac{m}{e} = \frac{(\mu_0 Hr)^2}{ER}$$

Table 1 lists the experimental values of u, the expected values of

$$\frac{m}{m_0} = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}$$

(where m_0 is the rest mass of the electron), and the observed values of the same quantity. We see that theory and experiment are in good agreement for three different groups of electrons with velocities upto 0·75c.

Table 1. Results of an Experiment on the Variation of the Relativistic Mass with Velocity

Electron velocity (in units of c)	Expected value of $\frac{m}{m_0} = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}$	Observed value of $\frac{m}{m_0}$
0·6337	1·293	1·293
0·6961	1·404	1·393
0·7496	1·507	1·511

7.2 Particle accelerators

Since 1939 the successful operation of large machines for accelerating sub-atomic particles to very high energy has shown, indirectly, that the formula for variation of relativistic mass with velocity must be correct for velocities very close indeed to c. The principle of the cyclotron, invented by Lawrence in 1931, is illustrated in Fig. 19. A short hollow metal cylinder divided into two 'dees' is placed between the poles of an electromagnet

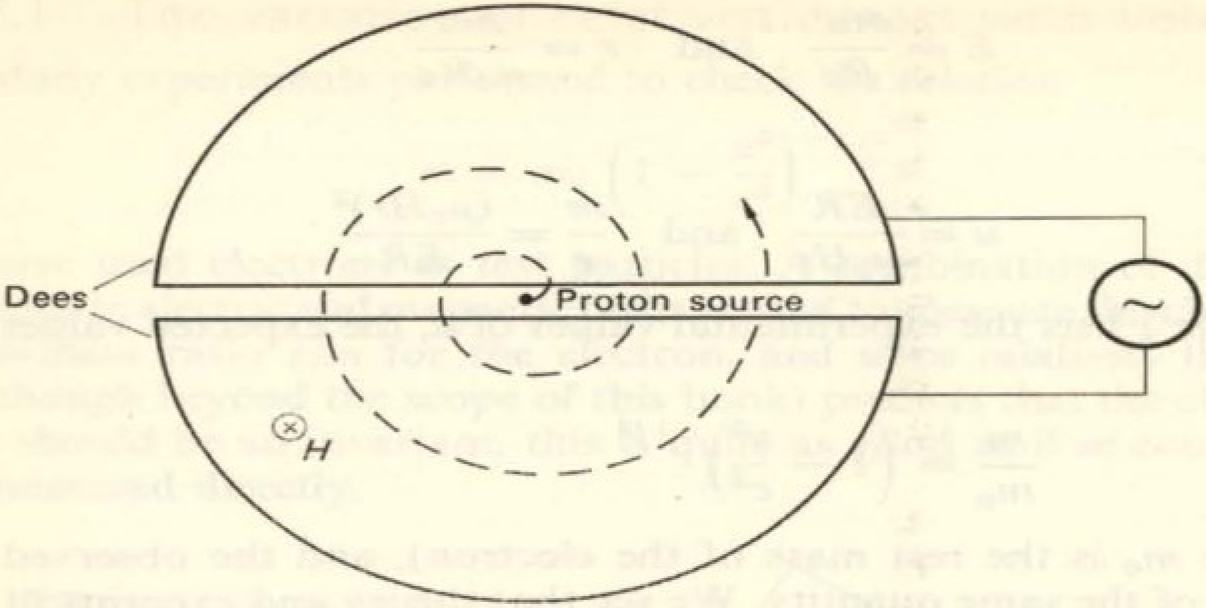


Fig. 19

providing a constant magnetic field H. Protons introduced at the centre of the dees by ionisation of hydrogen gas will travel in a circle of radius

$$r = \frac{mu}{\mu_0 He}$$

and may be accelerated by an electric field between the dees as they pass from one to the other. Continuous acceleration is obtained by using an a.c. field with a period of alternation equal to the time taken for the protons to complete one revolution. This period is given by

$$\frac{2\pi r}{u} = \frac{2\pi m}{\mu_0 He},$$

and is independent of the radius of the proton orbit, although the latter will increase with increasing energy. As long as u remains small, the frequency of the accelerating voltage remains constant, but as the velocity increases, this frequency must be varied during the acceleration to allow for the increase in relativistic mass of the protons.

The largest present-day (1970) accelerators are called proton synchrotrons. These operate on the same principle as the cyclotron except that both the frequency of the accelerating voltage

and the strength of the magnetic field are varied during the acceleration cycle. In this way the radius of the protons' orbit can be kept constant and the size of the magnet reduced. (This can be very important for financial considerations in machines which are several hundred metres in diameter.) The synchrotron at Serpukhov in the U.S.S.R. has recently accelerated protons to a total energy of seventy times their rest energy, which means that the formula for the relativistic mass must be correct up to a velocity of $0.9999c$. An even bigger machine is under construction at Weston, Illinois.

7.3 Particles, time dilation, and the second postulate

To compare theory with experiment in the decay of muons from cosmic rays, which was quoted in Chapter 3 as evidence for time dilation, we needed to know the velocity of the muons. This was taken to be $0{\cdot}99c$ with no indication of how the number was obtained. In fact, since the muons arrive from outer space at random times, the normal method of measuring velocity by recording the time of flight of the muons over a known distance is difficult to carry out, and an alternative procedure is used. This consists of measuring the total energy of the muons and comparing it with their rest mass to obtain their velocity as a fraction of c. Velocities of sub-atomic particles are often most conveniently measured in this way.

The effects of time dilation are continually being observed by people who work with particle accelerators. For instance, they are very important in making observations of short-lived elementary particles, with lifetimes of the order of 10^{-10} s. The apparatus for detecting these particles cannot be located nearer than a few metres to the point where they are produced. Classically, particles with such lifetimes would all decay in the first few centimetres, even if they were produced with velocities close to that of light. Fortunately, in practice, the time dilation factor is large and most of the particles reach the detector before decaying. (**Exercise.** If the particles are produced with rest mass

m_0, momentum p, and proper mean-life τ, and the detector is a distance L away, show that a fraction $\exp(-m_0L/p\tau)$ of the particles reach it before decaying. A proper mean-life of τ means that, at rest, a fraction $\exp(-t/\tau)$ of the original particles remain after time t.)

An experiment using the proton synchrotron at the European Centre for Nuclear Studies (C.E.R.N.) at Geneva, provides our most direct evidence in favour of the second of Einstein's postulates: the velocity of light is independent of the velocity of its source. Particles called π^0-mesons (or, very often, pi-zeros) were produced with total energy more than forty-five times their rest energy (i.e. velocities in excess of $0{\cdot}99975c$), and were allowed to decay into two photons or light 'particles'. The velocity of these photons was then measured in a careful time-of-flight experiment using fast electronics. (The accelerator produced pi-zeros in bunches at regular intervals, which made the timing easier.) The velocity of light obtained in this way agreed with ordinary measurements of c using light from stationary sources to better than one part in ten thousand!

7.4 Units of mass, energy, and momentum

The equivalence of mass and energy can be made more obvious if we measure both quantities in the same units. On an atomic scale, energies are conveniently measured in terms of the electronvolt (eV), which is the energy acquired when one electronic charge passes through a potential difference of one volt. The kilo- and mega-electronvolt (keV and MeV, equal to 10^3 and 10^6 electronvolts respectively, are also useful. One electronvolt is numerically equal to $1{\cdot}6 \times 10^{-19}$ Joules.

To measure masses in electronvolts, we use the equation

$$E = mc^2.$$

A mass of 1eV is equal to

$$\frac{1{\cdot}6 \times 10^{-19}}{9 \times 10^{16}} = 1{\cdot}8 \times 10^{-36}\ \text{kg},$$

which makes the rest-energy of the electron with rest mass 9×10^{-31} kg, equal to

$$\frac{9 \times 10^{-31}}{1{\cdot}8 \times 10^{-36}} = 5 \times 10^{5}\ \text{eV} \quad \text{or} \quad 0{\cdot}5\ \text{MeV}.$$

(Converting all the rest mass of one electron into energy would therefore produce $5 \times 10^{5} \times 1{\cdot}6 \times 10^{-19} = 8 \times 10^{-14}$ J.) To emphasise that the mass of the electron expressed in this way is a rest mass and not an energy we usually write it as $0{\cdot}5\ \text{MeV}/c^2$. The rest mass of a proton is $1{\cdot}6 \times 10^{-27}$kg or 938 MeV/c^2, which for many purposes can be approximated as 1000 MeV/c^2.

The equation $E^2 = p^2c^2 + m_0{}^2c^4$ shows that momentum has dimensions of energy divided by velocity. We can therefore measure momenta in units of MeV/c, where 1 MeV/c is equal to $(1{\cdot}6 \times 10^{-19})/(3 \times 10^{8}) = 5 \times 10^{-29}$ N m. If all energies are expressed in MeV, all momenta in MeV/c, and all masses in MeV/c^2, many of the factors of c in the formulae of relativistic mechanics disappear. For instance, equations (5.5), (5.6), and (5.7) then become

$$E^2 = p^2 + m_0{}^2, \quad \frac{p}{E} = \frac{u}{c}, \quad p^2 = T^2 + 2m_0T,$$

which is very convenient when doing complicated calculations.

7.5 Experiments on the equivalence of mass and energy

Nuclear masses can be measured very precisely by an instrument called a mass spectrograph, which finds the charge-to-mass ratio for the nucleus from deflections in electric and magnetic fields in comparison with similar ratios for nuclei of known mass. The results of the measurements are usually expressed in atomic mass units, each of which is equal to 1/12 of the mass of the C^{12} atom or $931{\cdot}44\ \text{MeV}/c^2$.

In atomic mass units the rest masses of the photon and Li^7

and He^4 nuclei are 1·007825 u, 7·016004 u, and 4·002603 u. The kinetic energy released in the reaction

$$p + Li^7 = He^4 + He^4$$

should therefore be equal to

$$(7{\cdot}016004 + 1{\cdot}007825 - 2 \times 4{\cdot}002603) \times 931{\cdot}44 = 17{\cdot}3\ \text{MeV}.$$

The kinetic energy of each α-particle has been measured as 8·6 MeV, in very good agreement with this prediction.

Photo-disintegration of deuterium occurs when a photon of sufficient energy strikes a deuterium nucleus (rest mass 2·014102 u) and makes it break up into a proton (1·007825 u) and a neutron (1·008665 u). It is found experimentally that photons of energy less than 2·225 ± 0·001 MeV fail to cause disintegration. From the masses quoted above this value should be

$$(1{\cdot}007825 + 1{\cdot}008665 - 2{\cdot}014102) \times 931{\cdot}44 = 2{\cdot}2246\ \text{MeV}.$$

These are only two examples of the verification of mass-energy equivalence in nuclear physics. In fact workers in this field are so convinced of its correctness that they find it easier to work out energy yields in nuclear reactions from accurate mass spectrographic measurements of the masses on both sides of the equation than to make direct measurements of the kinetic energy of the products of the reaction!

7.6 Particle and anti-particle production

One consequence of mass-energy equivalence which has no parallel in classical physics is the creation of particles in collision processes. When the accelerated beam of protons from a proton synchrotron collides with a hydrogen target, many reactions occur, among them the production of pi-zeros by the process

$$p + p \rightarrow p + p + \pi^0,$$

in which some 135 MeV of the incident proton's kinetic energy becomes converted into the rest energy of the pi-zero. Production processes of this kind are the normal way of using accelerators to make elementary particles.

Anti-particles can also be made in this way. An anti-particle has the same rest mass as the corresponding particle but the opposite sign of charge. For instance, the anti-particle of the electron, discovered by Anderson in 1932, is positively charged and called the positron. When a positron comes into contact with ordinary matter it is slowed down by collisions until it meets an electron and annihilation takes place, converting the rest mass of both particles into two or more photons. **(Exercise.** Why must there be at least two photons? If there are only two, what is the energy of each?)

After the discovery of the positron, interest centred on looking for the anti-proton. Because of its greater rest mass this particle can only be produced by incident protons of high energy. A proton and an anti-proton must be produced together in the reaction

$$\mathrm{p} + \mathrm{p} \rightarrow \mathrm{p} + \mathrm{p} + \mathrm{p} + \tilde{\mathrm{p}},$$

where $\tilde{\mathrm{p}}$ indicates the anti-proton. Fig. 20 shows the final state arrangement which requires least energy – that in which all

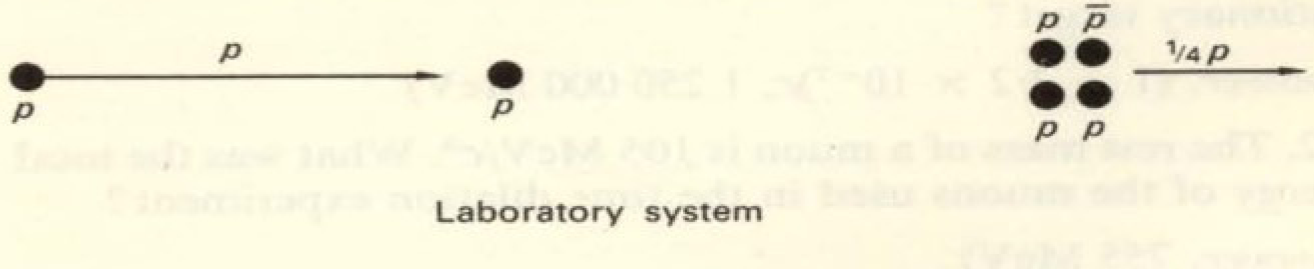


Fig. 20

four particles have the same laboratory momentum. (This means that in the centre-of-mass coordinate system they are all at rest.) Assuming that the incident proton has momentum p and kinetic energy T, the momentum of each of the final state particles must be $\frac{1}{4}p$. Energy conservation then gives

$$\sqrt{(p^2 + m_0{}^2)} + m_0 = 4\sqrt{(p^2/16 + m_0{}^2)},$$

where m_0 is the rest mass of a proton or anti-proton. (Factors of c have been omitted as we intend to use units of MeV and MeV/c^2.) Squaring up gives, after cancellations

$$2m_0(T + m_0) = 2m_0\sqrt{(p^2 + m_0^2)} = 14m_0,$$

from which

$$T = 6m_0 \approx 6000 \text{ MeV}.$$

A proton synchrotron (the Bevatron) was designed and built at the University of California expressly to accelerate protons to this energy, and as a result the anti-proton was discovered there by Segrè and others in 1958. When an anti-proton and a proton annihilate, 1876 MeV of rest energy is released and usually appears in the form of four or five of the particles called π-mesons.

Additional exercises

1. Two accelerators, producing protons of 25 000 MeV total energy, are set up facing each other so that the two beams meet head-on. Calculate the relative velocity between the protons in the two beams. What total energy would be necessary to produce the same relative velocity with one accelerator and a stationary target?

(*Answer*. $(1 - 3{\cdot}2 \times 10^{-7})c$, 1 250 000 MeV)

2. The rest mass of a muon is 105 MeV/c^2. What was the total energy of the muons used in the time dilation experiment?

(*Answer*. 755 MeV)

3. A pi-zero (rest mass 135 MeV) with total energy 1350 MeV decays into two photons which are emitted at equal angles to its direction of motion. What is the angle between the two photons?

(*Answer*. $2 \sin^{-1}(0{\cdot}1)$ or $11{\cdot}5°$)

4. Use the masses quoted in the text to calculate how many kilowatt-hours of power could be supplied from the conversion

of one mole (6×10^{23} atoms) of deuterium into helium by the reaction $H^2 + H^2 \rightarrow He^4$.

(*Answer*. $3{\cdot}2 \times 10^5$)

5. The rest mass of the pi-zero is 135 MeV. What is the least proton kinetic energy for its production in the reaction $p + p \rightarrow p + p + \pi^0$?

(*Answer*. $(2m_p + m_{\pi^0}^2/2m_p)c^2$ or 280 MeV. In this case all the final state particles are at rest in the centre-of-mass coordinate system.)

6. A proton and an anti-proton annihilate at rest in the laboratory into two pi-zeros. What is their velocity?

(*Answer*. $0{\cdot}9895c$)

8

General Relativity

The basic principle of Special Relativity is that all inertial coordinate systems are equally good for the description of nature, and hence the laws of physics must be invariant under Lorentz transformations. Although we have obtained many important results in the last few chapters, the restriction to one particular kind of coordinate system has sometimes proved inconvenient: for example our solution of the Twin Paradox in Chapter 5 applied only to one particular method of making the journey, which had been specially selected to avoid the use of accelerated coordinate systems. In this chapter we shall see in outline how the principles of Relativity may be extended to deal with observations in any kind of coordinate system.

8.1 The principle of Equivalence

There are two kinds of mass in classical physics, inertial masses, which are inversely proportional to the acceleration produced by a known force, and gravitational masses, which are directly proportional to the force exerted by a known gravitational field. For example, in the usual equation expressing Newton's second law of motion for a particle in a gravitational field

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = m\,g, \tag{8.1}$$

the m on the left is the inertial mass of the particle and that on the right is its gravitational mass. Experiments by Eōtvos (1922) and Dicke (1964) have shown that the inertial and gravitational masses of the same body are numerically equal to about 3 parts in 10^{11}. It follows from equation (8.1) that the acceleration

produced by a gravitational field is independent of the mass of the particle.

Einstein saw that the relation between inertial and gravitational masses is deeper than a mere numerical equality. Consider two observers in identical spaceships, O at rest on earth and O′ in an empty region of space but accelerating with an acceleration g. How can O and O′ distinguish between their two situations (assuming that looking out of the windows is forbidden)? If O drops something, it will fall to the floor with an acceleration of $9{\cdot}81$ m/s^2. But so will any object dropped by O′. Suppose O times the swings of a simple pendulum to find a value for the 'acceleration due to gravity' in his vicinity: O′ will get exactly the same answer from a similar experiment. It is impossible to think of any mechanics experiment which will distinguish between them. Of course if O moves around he will detect small effects due to the fact that the earth's gravitational field is radial, whereas O′s acceleration is in one straight line. For instance, two objects dropped by O, one from each hand, will draw slightly closer together as they fall, but a similar pair dropped by O′ will always remain the same distance apart. This does not alter the essential point however, which is that at any one point a distinction by mechanical means between gravitational and inertial forces is impossible.

Einstein extended this conclusion from mechanics to apply to all kinds of physical phenomena. His postulate, known as the *Principle of Equivalence*, states: *there is no way whatever of distinguishing gravitational forces from those due to accelerated motion*. An obvious consequence is that inertial coordinate systems are no longer in any sense unique. From the point of view General Relativity, all coordinate systems, accelerated or not, are equally appropriate for the description of nature. Experiments to verify this postulate will be discussed in a later section.

8.2 Space and General Relativity

Although the Principle of Equivalence can be stated so simply, to make deductions from it we often need rather complicated

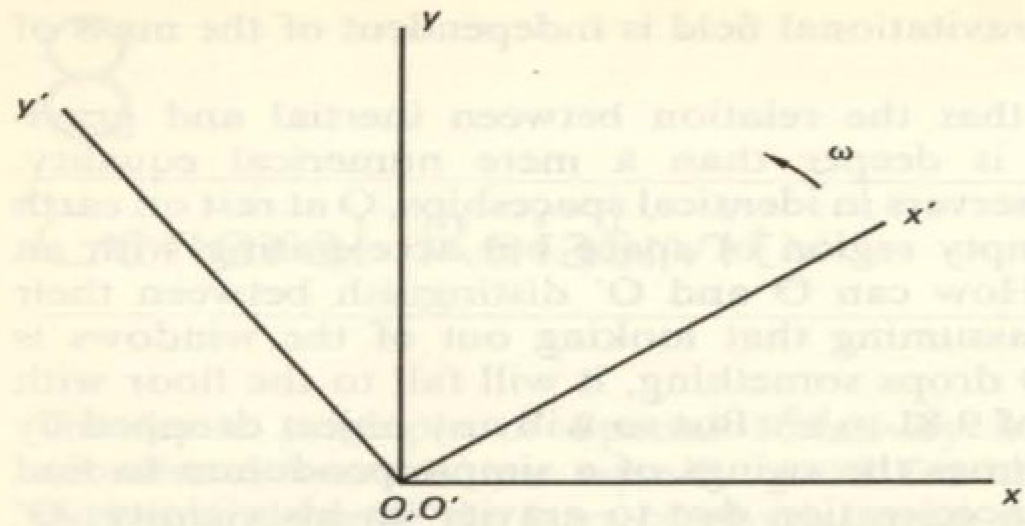


Fig. 21

mathematics. A simple example will reveal some of the difficulties involved. Let S and S' be two coordinate systems with Oz and $O'z'$ axes coincident, as in Fig. 21. A circle of radius r is drawn around the origin in the $x'y'$ plane. To determine the ratio of the circumference of this circle to a diameter, we can take a large number of measuring rods of equal length and lay them end to end round the circle and across it. If S' does not rotate, the number of rods round the circumference n_c, and the number across the diameter n_d, will be related by

$$\frac{n_c}{n_d} = \pi,$$

at any rate in the limit as all the rods become very short.

But if S' rotates around Oz' with angular velocity ω, the ratio will change. Even if the rods are so rigid that the accelerations involved do not affect their length, or more realistically, if we are able to calculate this deformation and make allowance for it, we must still consider effects due to the different velocities of the rods. In S, the length of the rods along the diameter will not be affected, since their velocity is perpendicular to their length, but the rods around the circumference will all be contracted by the usual length contraction factor, which is

$\sqrt{(1 - \omega^2r^2/c^2)}$ in this case. More rods will therefore be needed to complete the circumference, making the ratio

$$\frac{n_c}{n_d} > \pi$$

(and dependent on the radius of the circle).

Such a result is impossible in a space where the usual rules of Euclidean geometry apply. But it is easy to find kinds of space where the geometry is non-Euclidean. Consider the two-dimensional surface of a sphere in ordinary three-dimensional space. A two-dimensional geometer, confined to this space and unable to move out of it, would still have ways of deciding that his two-dimensional space obeyed a non-Euclidean geometry. For instance, the angles of a triangle drawn on his surface do not add up to 180°. In the same way the anomalous value of the ratio of circumference to diameter in our hypothetical experiment shows that, if the Principle of Equivalence is true, our normal three-dimensional space is really a curved surface in a Euclidean space of some higher number of dimensions. The detailed mathematics is very complicated, and time must be included as well as space, but it turns out that the usual four variables (x, y, z, t) in fact represent a four-dimensional curved surface in a ten-dimensional Euclidean space-time. The curvature at each point of this space is a function of the gravitational field there, i.e. the geometry of space-time depends on the amount of matter present. No wonder calculations in General Relativity are very difficult!

8.3 The gravitational red shift

The detailed predictions of General Relativity are nothing like so well understood and experimentally verified as those of Special Relativity. This is partly because of the mathematical difficulties and partly because of the extreme difficulty in doing sufficiently accurate experiments. The existence of a gravitational red shift is one of the few effects which is fairly easy to calculate and which can be tested experimentally.

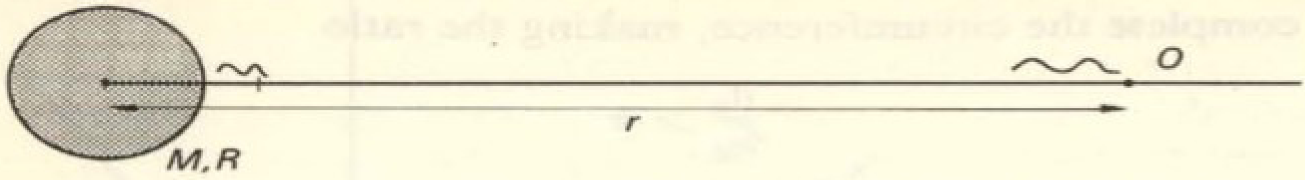


Fig. 22

Fig. 22 illustrates the emission of a photon of frequency ν from the surface of a star of mass M and radius R. The frequency of the photon is measured by an observer at a distance r from the centre of the star. What value does he obtain? We can find the answer by considering the energy of the photon. The potential energy of the photon increases by

$$\frac{h\nu}{c^2}\left(\frac{GM}{R}-\frac{GM}{r}\right),$$

where $h\nu/c^2$ is the mass of the photon and G the gravitational constant. This must be compensated by a decrease in the photon's energy of motion $h\nu$, i.e. by a shift of frequency $\Delta\nu$ given by

$$\frac{\Delta\nu}{\nu}=\frac{-GM}{c^2}\left(\frac{1}{R}-\frac{1}{r}\right)=\frac{-\Delta\phi}{c^2}.$$

Here $\Delta\phi$ is the change in gravitational potential, i.e. the work done in taking unit mass from the surface to a distance r. Hence light is shifted towards the red end of the spectrum (decreasing ν) as it climbs the potential gradient away from the star.

This red shift can be detected using the Fraunhöfer lines in the spectrum of the light emitted from stars. For the sun, $\Delta\nu/\nu$ is about -2×10^{-6}, but for the superdense white dwarf stars, such as Sirius, it can be as large as 300 times this value. Astronomical measurements of the red shift agree with theory to about 5%, although there are sometimes difficulties in interpreting the measurements.

Terrestrial experiments, though less open to differences of interpretation, are very difficult to carry out because of the small size of the effect. Over vertical distances of the order of

30 m near the surface of the earth the expected shift is only a few parts in 10^{15}. Fortunately an effect discovered by Mossbauer in 1958 provided a way of obtaining certain photons (γ-rays emitted in the decay of Fe^{57} nuclei) with a very constant frequency, which could be used in red shift experiments. In the experiment of Pound and Rebka (1964) photons emitted at the bottom of a tower were red shifted before they reached a detector at the top. By compensating the gravitational red shift with the Doppler effect obtained by moving the detector at a velocity of a few mm/s, measurements were made confirming the theory to about 10%.

8.4 Time measurements in General Relativity

An atom emitting a spectral line may be regarded as a clock, with each period of the light wave indicating a time interval of $1/\nu$. The conclusions of the last section will therefore apply equally to any form of time-indicating device. They show that an observer looking at a clock at a lower (gravitational) potential will observe that clock to run slow relative to his own clock. In a gravitational field the effect is reversible: i.e. light travelling towards the star will be increased in frequency or shifted towards the blue end of the spectrum by the same amount as that in which light travelling in the other direction is shifted towards the red.

Let us use these results in a consideration of time measurements in an accelerated coordinate system. The observer O in Fig. 23 is at rest in an inertial coordinate system, while O′

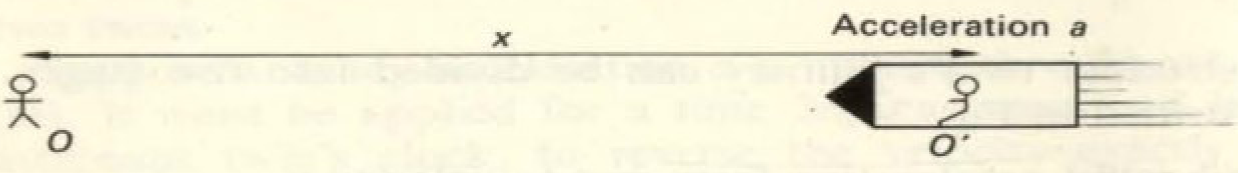


Fig. 23

moves towards him with acceleration a. By the Principle of Equivalence, O′ can consider himself to be in an inertial system with a uniform gravitational field a exerted in all the surrounding

space. When his distance from O is x he will judge the gravitational potential difference between them to be

$$\Delta\phi = ax,$$

and will expect O's clock to be running faster than his own. In fact a time interval Δt on O's clock will be related to the corresponding interval $\Delta t'$ on O′'s clock by

$$\frac{\Delta t'}{\Delta t} = \frac{\nu + \Delta\nu}{\nu} = 1 + \frac{ax}{c^2}. \tag{8.2}$$

When O′ passes O and accelerates away from him, the sign of $\Delta\phi$ will change and O′ will see O's clock to be running slow.

O sees the situation very differently. He does not accelerate and so does not experience any of the effects observed by O′. He will indeed see O′'s clock to run slow, but only by the usual Special Relativity time dilation factor, which depends on the velocity of O′ but not on his acceleration. (Of course O′ also observes the Special Relativity time dilation due to the relative velocity between himself and O. The point is that only O′ experiences the effect due to their relative acceleration.) We shall now use these ideas to present a fuller explanation of the original form of the Twin Paradox.

8.5 Another solution of the twin paradox

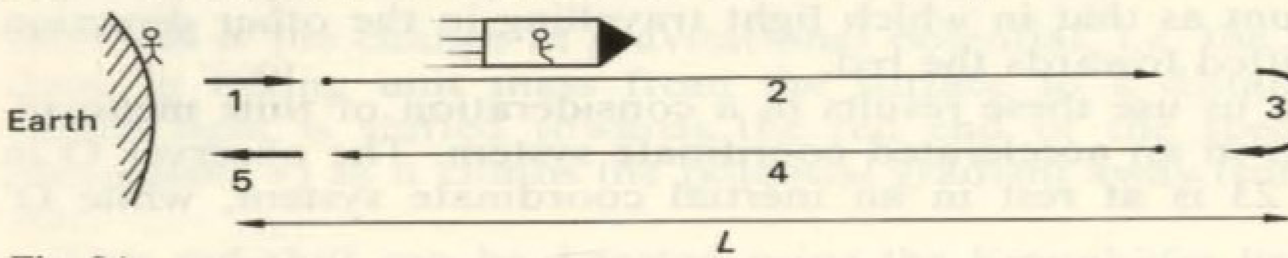


Fig. 24

The astronaut twin's journey can be divided into five stages, as illustrated in Fig. 24:

(i) rapid acceleration from rest to velocity v,
(ii) steady motion at velocity v for a distance L,
(iii) rapid deceleration to change the velocity to v in the other direction,
(iv) steady motion at velocity v for a distance L,
(v) rapid deceleration from velocity v to rest.

From the earth twin's point of view, stages (ii) and (iv) dominate the journey. His clock records the total time for the journey as

$$\frac{2L}{v}$$

and, allowing for time dilation, he expects the astronaut twin's clock to record

$$\frac{2L}{v}\sqrt{\left(1-\frac{v^2}{c^2}\right)}.$$

Thus the astronaut twin should return the younger by a time

$$\frac{2L}{v}\left\{1-\sqrt{\left(1-\frac{v^2}{c^2}\right)}\right\}\approx\frac{Lv}{c^2}\text{ if }v\ll c.$$

The earth twin observes no effects due to the accelerations experienced by the astronaut twin in stages (i), (ii) and (v).

The astronaut twin expects the earth twin's clock to run slow during stages (ii) and (iv) of his journey. If no other stages were involved he would expect the earth twin to be the younger by a time Lv/c^2 when they meet again. The paradox seems to have re-appeared, but it is only because we have ignored the accelerations. The effects of the accelerations in stages (i) and (v) are exact opposites and so compensate each other (as do the effects of the earth's gravitational field), but during the whole of stage (iii) the astronaut is accelerated towards the earth. This produces a permanent time difference between the clocks of the two twins.

Suppose a steady acceleration a is applied throughout stage (iii). It must be applied for a time $2v/a$, as measured by the astronaut twin's clock, to reverse the velocity exactly. The potential difference between the twins is then aL, so, using equation (8.2), the corresponding time interval on the earth twin's clock will be

$$\frac{2v}{a}\left(1+\frac{aL}{c^2}\right)=\frac{2v}{a}+\frac{2Lv}{c^2}.$$

The extra time $2Lv/c^2$ is just enough to cancel and reverse the effect of stages (ii) and (iv), and to ensure that both twins agree that the astronaut twin is younger by a time Lv/c^2 when they meet again.

To summarise, the earth twin observes the astronaut twin's clock to run slow throughout his journey. The astronaut twin observes the earth twin's clock to run slow for most of the journey, except for stage (iii), when it runs ahead very fast. The total effect is just sufficient to remove the paradox. This solution is only exact to first order but a more detailed calculation would show that the removal of the paradox is exact, as expected.

Additional exercises

1. The white dwarf star Sirius has a mass of 2×10^{30} kg and a gravitational red shift of 0·0007. What is its density? (Gravitational constant $G = 6{\cdot}7 \times 10^{-11}$ N m²/kg²)

(*Answer*. 5×10^{10} kg/m³)

2. In the experiment by Pound and Rebka, the absorber and detector were separated by 72 ft. What was the theoretical value of the red shift? (Gravitational acceleration at the surface of the earth $g = 9{\cdot}8$ m/s², 1 m = 3·28 ft)

(*Answer*. $2{\cdot}5 \times 10^{-15}$)

3. A satellite in orbit 100 km above the earth has a period of 90 minutes. How well does an observer on earth expect a clock on the satellite to keep time? (Take g as 9·8 m/s² and ignore its variation with height. Radius of the earth = 6400 km)

(*Answer*. The clock runs slow by 6×10^{-6} s per orbit because of the gravitational red shift. It also loses $1{\cdot}7 \times 10^{-6}$ s per orbit because of time dilation.)

